Petri Nets

Sources

Gang Quan

Review

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  - NFSM
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Petri Net

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Introduction

- Originated from Carl Adam Petri’s dissertation in 1962
- A graphical and mathematical modeling tool
- An effective and promising tool for capturing system concurrent, asynchronous, distributed, parallel, nondeterministic, stochastic characteristics.
- A bridge between the practitioners and theoreticians
- Various applications:
  - Performance evaluation
  - System verification
  - Communication protocols
  - Distributed database, etc
What is a Petri Net

- A directed, weighted, bipartite graph
- \( G = (V,E) \)
  - **Nodes** (\( V \))
    - places (shown as circles)
    - transitions (shown as bars)
  - **Arcs** (\( E \))
    - from a place to a transition or from a transition to a place
    - labeled with a weight (a positive integer, omitted if it is 1)
What is a Petri Net (Cont’d)

- **Marking (M)**
  - An $m$-vector $(k_0, k_1, \ldots, k_m)$
    - $m$: the number of places
    - $k_i \geq 0$: the number of “tokens” in place $p_i$

- **Modeling**
  - Places $\leftrightarrow$ Input/output data
  - Transitions $\leftrightarrow$ Computation
  - Arcs (E)
    - Place $\rightarrow$ Transition: consume input data
    - Transition $\rightarrow$ place: produce output data
Firing rules
- An enabled transition
- An enabled transition may or may not fire
- A firing of an enabled transition
An Example

- $2H_2 + O_2 \rightarrow 2H_2O$
Formal Definition of Petri Net

- **PN = ( P, T, F, W, M₀)**
  - \( P = \{p₀,p₁,…,pₘ\} \): a finite set of places
  - \( T = \{t₁,t₂,…,tₙ\} \): a finite set of transitions
  - \( F \subseteq (P \times T) \cup (T \times P) \): a set of arcs (flow relation)
  - \( W: F \rightarrow \{1,2,3,…\} \) weight function
  - \( M₀: P \rightarrow \{0,1,2,…\} \) initial marking
  - \( P \cap T = \emptyset \quad P \cup T \neq \emptyset \)
Formal Definition of Petri Net (cont’d)

Places:
Transitions:
Arcs:
Weight:
Initial marking:
Formal Definition of Petri Net (Cont’d)

- Source transition and sink transition

- Pure petri net and ordinary petri net
  - Pure petri net: no self loop
  - Ordinary petri net: all the weights are 1’s.

- Infinite/finite capacity petri net
  - K(p)
Formal Definition of Petri Net (Cont’d)

- **Strict/weak transition rule**
  - Strict:  $K(p) < \infty$
  - Weak:  $K(p) = \infty$

- **Theorem:**
  - For any pure finite-capacity net $(N,M_0)$ with a strict transition rule, there must be another equivalent infinite-capacity net $(N’,M’_0)$ with a weak transition rule.
Equivalence

- **Reachability Graph** $G = (V, E)$
  - $V$: markings
  - $E$: firings

- **Equivalence**
  - Two petri nets $(N, M_0)$ and $(N', M'_0)$ are equivalent iff for any possible firing sequence in $(N, M_0)$ same firing sequence can be found in $(N', M'_0)$ and vice versa.
An Example

(a)

(b)

(c)
Parallel Activity

Sequencing

Synchronization
Parallel Activity (Cont’d)

Concurrence

Choice (conflict)

Confusion (conflict + currency)
Finite State Machine

- A vender machine
  - accepts 5, 10 cents
  - sell candy bars worth 15 or 20 cents
  - maximum hold up coins = 20 cents
Data Flow Graph

\[ X = \frac{a+b}{a-b} \]
Communication Protocol
Synchronization Control

- \( k \) processes
- More than two can read simultaneously
- One is writing, no one can read
- One is reading, no one can write
Multiprocessor Systems

- 5 processors
- 2 buses
- 3 memories

P1: running processes
P2: available buses
P3: access requests
P4: access to the shared memory
P5: processors requesting the same shared memory with that in P4
Properties

- Behavioral Properties
  - Properties hold only for the given initial marking

- Structural Properties
  - Independent of the initial marking
  - Depend on the topological structure of the nets
Behavioral Properties

- Reachability
- Boundedness
- Liveness
- Reversibility
- Coverability
- Persistence
- Synchronic distance
- Fairness
Reachability

- A Marking $M_n$ is **reachable** from marking $M_0$ if there exists a sequence of firings $\sigma = t_1 \ t_2 \ldots \ t_n$ that transforms $M_0$ to $M_n$.

$$M_0 = (1,0,1,0) \ x \ M_n = (1,1,0,0)$$

- $R(M_0)$: all the reachable markings
The net is \textit{safe} if 1-bounded
Liveness

- From any reachable marking any transition can become fireable

![Live Petri Net](image1)
![Nonlive Petri Net](image2)

- Different levels of live: dead, L1-live, L2-live, L3-live, L4-live
Reversibility

- For any $M_k$ in $R(M_n)$, $M_n$ is also in $R(M_k)$
  - $M_n$: *home state*
Coverability

- Exist marking $M'$ in $R(M_0)$ such that $M'(p_i) \geq M(p_i)$
  - Closely related to liveness
    - E.g. Let $M$ be the minimum marking needed to enable a transition $t$, then
      - $t$ is dead $\iff$ $M$ is not coverable
        - $t$ can never be fired since no enough tokens are available
      - $t$ is L1-live $\iff$ $M$ is coverable
        - $t$ can be at least fired once since $M'$ is reachable and can offer enough tokens
For any pair of enabled transitions, firing one will not disable another.

All the marked graphs are persistent but not vice versa.

- Marked graph: each place has single input and single output.
Persistence (Cont’d)

Persistent, but not marked graph!
The maximum possible difference between the numbers of times that two transitions fired

\[ d_{12} = \max_{\sigma} | T(t1) - T(t2) | \]

- \( \sigma \): a firing sequence starting from any marking \( M_n \) in \( R(M_0) \)
- \( T(t) \): the number of times that \( t \) is fired in \( \sigma \)

\[ d_{12} = 1 \]
\[ d_{34} = 1 \]
\[ d_{13} = \infty \]
Unconditional fairness

- A firing sequence is unconditional fair if every transition in the net can appear infinitely often.
- \((N,M_0)\) is an unconditionally fair net if every firing sequence starting from \(M\) in \(R(M_0)\) is unconditionally fair.
Properties

• Behavioral Properties
  – Properties hold only for the given initial marking

• Structural Properties
  – Independent of the initial marking
  – Depend on the topological structure of the nets
Structural Properties

- Structurally live
  - There exists a live initial marking for N
- Controllability
  - Any marking is reachable for any other marking
- Structural Boundedness
  - Bounded for any finite initial marking
Conservativeness

- The total number of the tokens in the net is a constant.
Structural Properties

- **Repetitiveness**
  - There exists a initial marking $M_0$ and a firing sequence from $M_0$ such that each transition can occur infinitely often.

- **Consistency**
  - There exists a initial marking $M_0$ and a firing sequence from $M_0$ back to $M_0$ such that each transition occurs at least once.
Petri Extensions

● Timed petri net
  – Introduce time delays associated with transitions and/or places
    ● Deterministic net
      – Delays are determined
    ● Stochastic net
      – Delays are probabilistic

● Colored petri net
  – Tokens have different values (colors)
Summary

- Graphical interpretation
- Convenient for represent distributed, concurrency, synchronization, etc
- Properties
  - Behavioral
  - Structural
- Extensions