



Introduction to Quantum logic (2)

2002 .10 .10
Yong-woo Choi



Classical Logic Circuits

- ❖ Circuit behavior is governed implicitly

by classical physics

- ❖ Signal states are simple bit vectors,

e.g. $X = 01010111$

- ❖ Operations are defined by Boolean Algebra



Quantum Logic Circuits

- ❖ Circuit behavior is governed explicitly by quantum mechanics
- ❖ Signal states are vectors interpreted as a superposition of binary “qubit” vectors with complex-number coefficients

$$|\Psi\rangle = \sum_{i=0}^{2^n-1} c_i |i_{n-1}i_{n-1}\dots i_0\rangle$$

- ❖ Operations are defined by linear algebra over Hilbert Space and can be represented by unitary matrices with complex elements



Signal state (one qubit)

 $|0\rangle$

Corresponds to

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

 $|1\rangle$

Corresponds to

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

 $\alpha_0|0\rangle + \alpha_1|1\rangle$

Corresponds to

$$\alpha_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}$$

$$|\alpha_0|^2 + |\alpha_1|^2 = 1$$

More than one qubit

- ❖ If we concatenate two qubits

$$(\alpha_0|0\rangle + \alpha_1|1\rangle) (\beta_0|0\rangle + \beta_1|1\rangle)$$

$$\begin{aligned} |\alpha_0|^2 + |\alpha_1|^2 &= 1 \\ |\beta_0|^2 + |\beta_1|^2 &= 1 \end{aligned}$$

We have a 2-qubit system with 4 basis states

$$|0\rangle|0\rangle = |00\rangle \quad |0\rangle|1\rangle = |01\rangle \quad |1\rangle|0\rangle = |10\rangle \quad |1\rangle|1\rangle = |11\rangle$$

And we can also describe the state as

$$\alpha_0\beta_0|00\rangle + \alpha_0\beta_1|01\rangle + \alpha_1\beta_0|10\rangle + \alpha_1\beta_1|11\rangle$$

Or by the vector

$$\begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} \otimes \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} \alpha_0 \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} \\ \alpha_1 \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \alpha_0\beta_0 \\ \alpha_0\beta_1 \\ \alpha_1\beta_0 \\ \alpha_1\beta_1 \end{pmatrix}$$



Tensor product



Quantum Operations

❖ Any linear operation that takes states

$$\alpha_0|0\rangle + \alpha_1|1\rangle \quad \text{satisfying} \quad |\alpha_0|^2 + |\alpha_1|^2 = 1$$

and maps them to states

$$\beta_0|0\rangle + \beta_1|1\rangle \quad \text{satisfying} \quad |\beta_0|^2 + |\beta_1|^2 = 1$$

must be UNITARY

$$\overline{U}^t U = I$$

$$\Leftrightarrow \overline{U}^t = U^{-1}$$



Find the quantum gate(operation)

$$\overline{U}^t U = I$$

$$\Leftrightarrow \overline{U}^t = U^{-1}$$

From upper statement

We now know the necessities

1. A matrix must has inverse , that is **reversible**.
2. Inverse matrix is the same as \overline{U}^t .

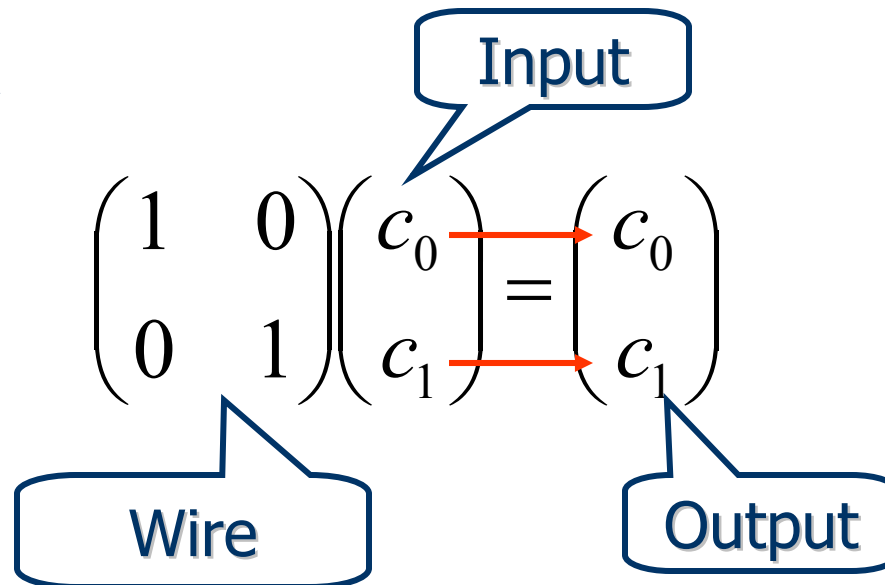
So, reversible matrix is good candidate for quantum gate

Quantum Gate

❖ One-Input gate: Wire

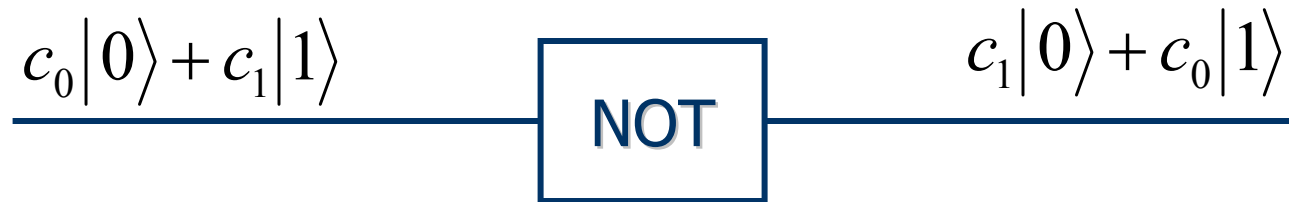


❖ By matrix

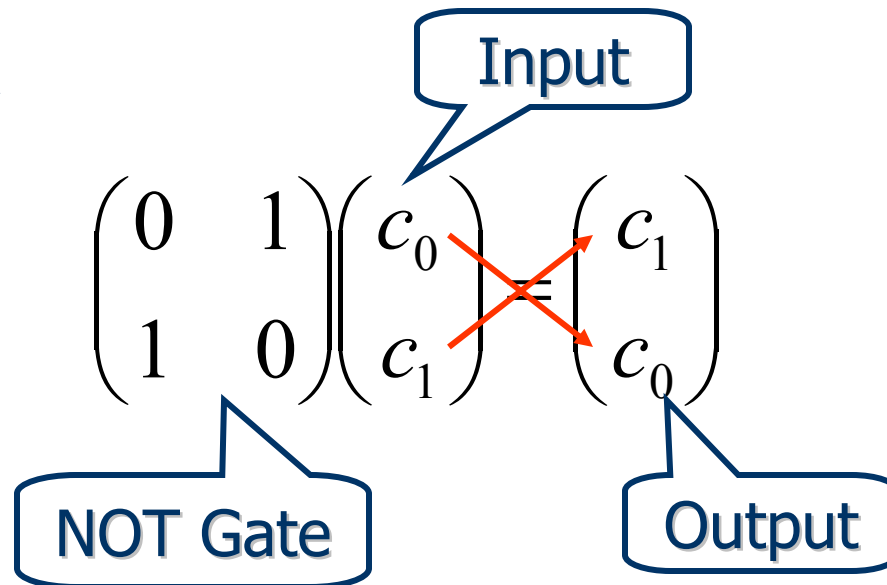


Quantum Gate

❖ One-Input gate: NOT

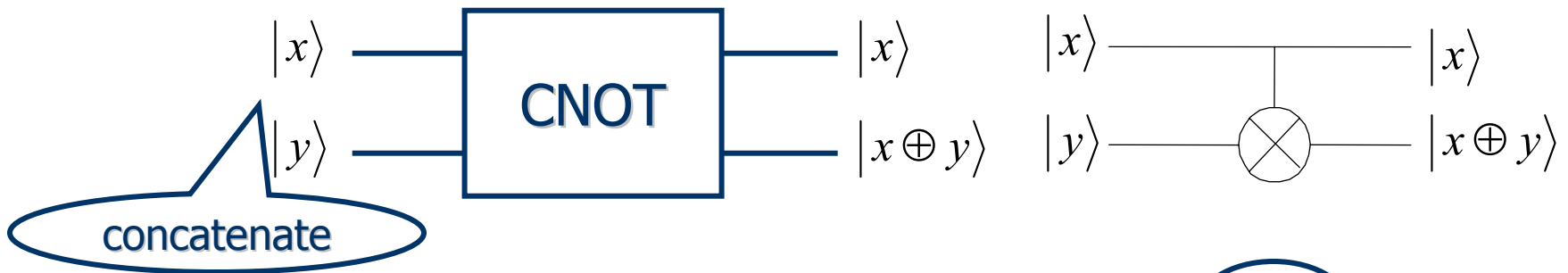


❖ By matrix



Quantum Gate

❖ Two-Input Gate: Controlled NOT (Feynman gate)



❖ By matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 00 \\ 01 \\ 10 \\ 11 \end{pmatrix} = \begin{pmatrix} 00 \\ 01 \\ 11 \\ 10 \end{pmatrix}$$

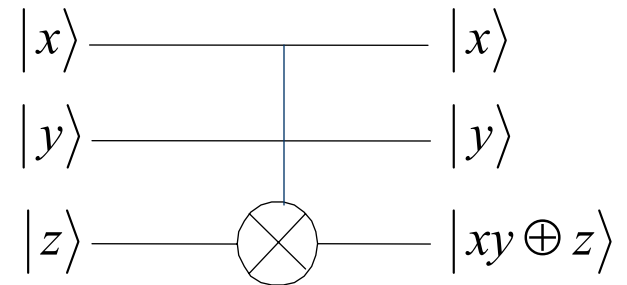
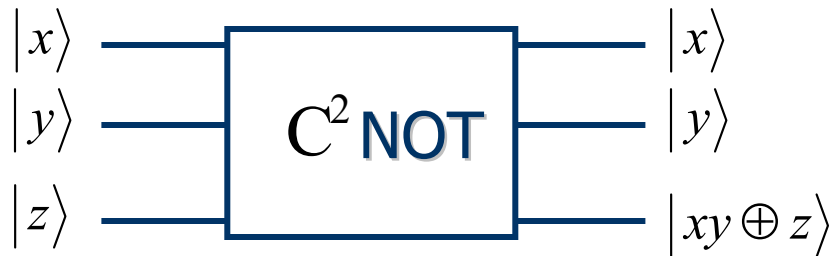
The matrix multiplication is shown with red arrows indicating the mapping from the input basis states to the output basis states: 10 maps to 11 and 11 maps to 10.

y

$$\begin{aligned}
 |x\rangle|0\rangle &\rightarrow |x\rangle|x\rangle \\
 |x\rangle|1\rangle &\rightarrow |x\rangle|\bar{x}\rangle
 \end{aligned}$$

Quantum Gate

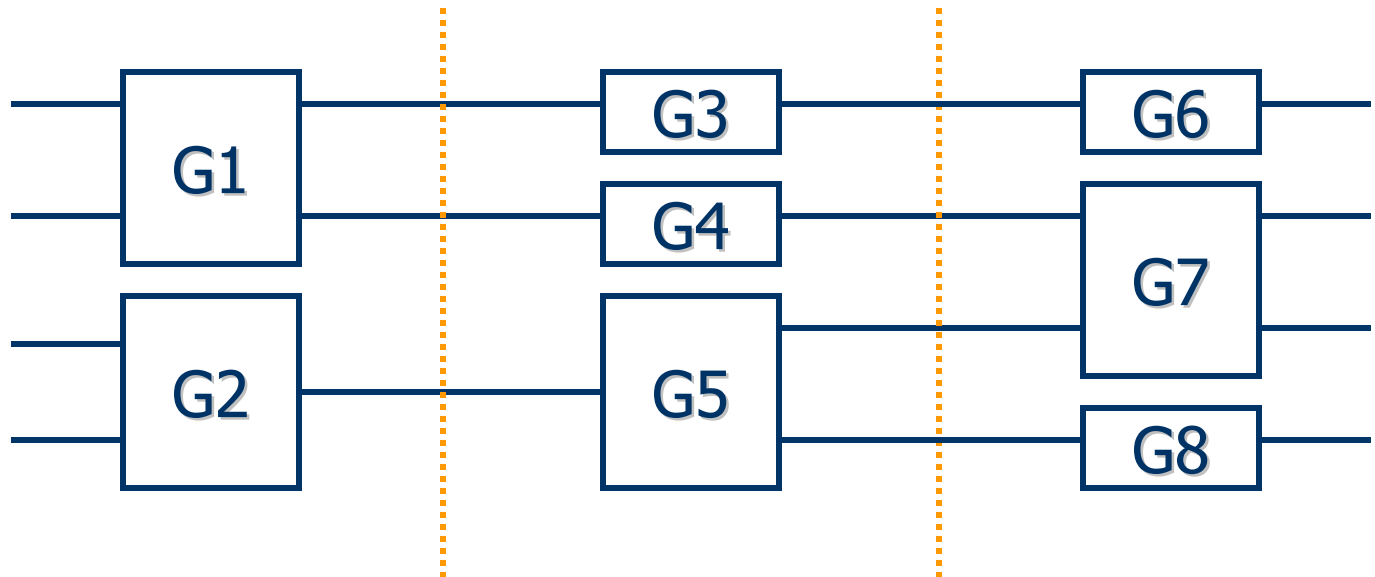
❖ 3-Input gate: Controlled CNOT (Toffoli gate)



❖ By matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Quantum circuit



Tensor
Product

$$G1 \otimes G2$$

$$G3 \otimes G4 \otimes G5$$

$$G6 \otimes G7 \otimes G8$$

Matrix
multiplication

$$(G6 \otimes G7 \otimes G8) * (G3 \otimes G4 \otimes G5) * (G1 \otimes G2)$$