Introduction to Quantum logic (2)

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Classical Logic Circuits

- Circuit behavior is governed implicitly

  by classical physics

- Signal states are simple bit vectors,

  e.g. X = 01010111

- Operations are defined by Boolean Algebra

Quantum Logic Circuits

- Circuit behavior is governed explicitly by quantum mechanics
- Signal states are vectors interpreted as a superposition of binary "qubit" vectors with complex-number coefficients

$$|\Psi\rangle = \sum_{i=0}^{2^n-1} c_i |i_{n-1} i_{n-1} \ldots i_0\rangle$$

- Operations are defined by linear algebra over Hilbert Space and can be represented by unitary matrices with complex elements

Signal state (one qubit)

$|0\rangle$ Corresponsd to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$|1\rangle$ Corresponsd to $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\alpha_0|0\rangle + \alpha_1|1\rangle$ Corresponsd to $\alpha_0\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha_1\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}$

$$|\alpha_0|^2 + |\alpha_1|^2 = 1$$

More than one qubit

- If we concatenate two qubits

$$\left(\alpha_0|0\rangle + \alpha_1|1\rangle\right)\left(\beta_0|0\rangle + \beta_1|1\rangle\right)$$

$$|\alpha_0|^2 + |\alpha_1|^2 = 1$$
$$|\beta_0|^2 + |\beta_1|^2 = 1$$

We have a 2-qubit system with 4 basis states

$$|0\rangle|0\rangle = |00\rangle \qquad |0\rangle|1\rangle = |01\rangle \qquad |1\rangle|0\rangle = |10\rangle \qquad |1\rangle|1\rangle = |11\rangle$$

And we can also describe the state as

$$\alpha_0\beta_0|00\rangle + \alpha_0\beta_1|01\rangle + \alpha_1\beta_0|10\rangle + \alpha_1\beta_1|11\rangle$$

Or by the vector

$$\begin{pmatrix}\alpha_0\\ \alpha_1\end{pmatrix} \otimes \begin{pmatrix}\beta_0\\ \beta_1\end{pmatrix} = \begin{pmatrix}\alpha_0\begin{pmatrix}\beta_0\\ \beta_1\end{pmatrix}\\ \alpha_1\begin{pmatrix}\beta_0\\ \beta_1\end{pmatrix}\end{pmatrix} = \begin{pmatrix}\alpha_0\beta_0\\ \alpha_0\beta_1\\ \alpha_1\beta_0\\ \alpha_1\beta_1\end{pmatrix}$$

[illegible]

Tensor product

Quantum Operations

- Any linear operation that takes states

$$\alpha_0|0\rangle + \alpha_1|1\rangle \quad \text{satisfying} \quad |\alpha_0|^2 + |\alpha_1|^2 = 1$$

and maps them to states

$$\beta_0|0\rangle + \beta_1|1\rangle \quad \text{satisfying} \quad |\beta_0|^2 + |\beta_1|^2 = 1$$

must be UNITARY

$$\overline{U}^t U = I$$

$$\Leftrightarrow \overline{U}^t = U^{-1}$$

Find the quantum gate(operation)

$$\overline{U}^t U = I$$

$$\Leftrightarrow \overline{U}^t = U^{-1}$$

From upper statement

We now know the necessities

1. A matrix must has inverse , that is reversible.

2. Inverse matrix is the same as $\overline{U}^t$.

So, reversible matrix is good candidate for quantum gate

Quantum Gate

- **One-Input gate:** Wire

- **By matrix**

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} c_0 \\ c_1 \end{pmatrix}$$

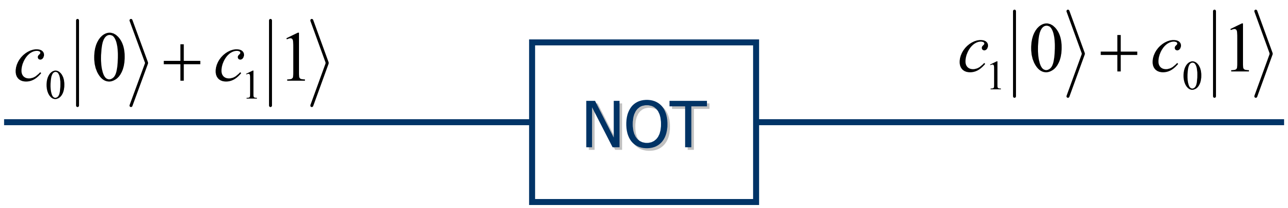
Input

Wire

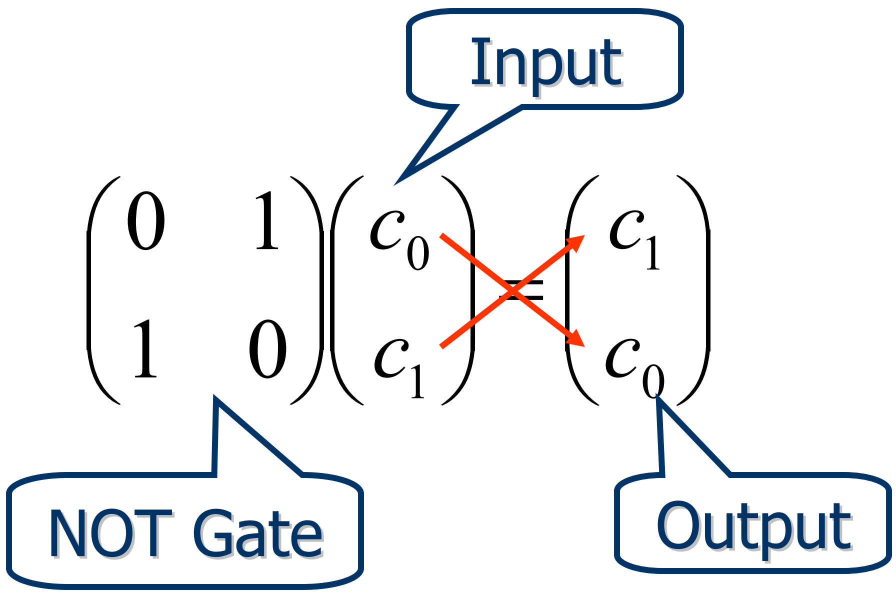
Output

Quantum Gate

- **One-Input gate:** NOT

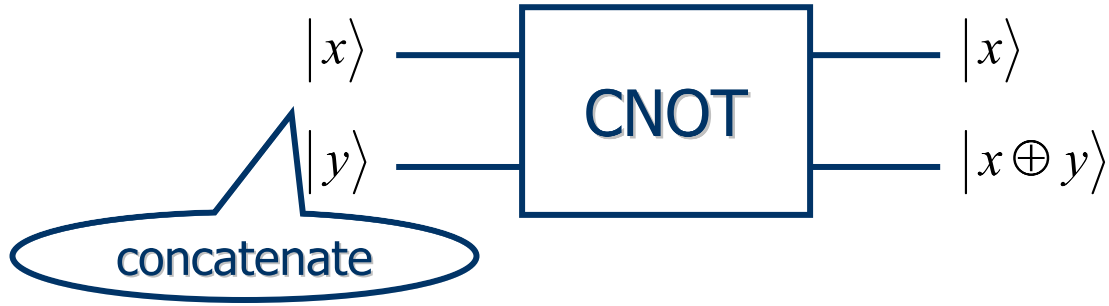


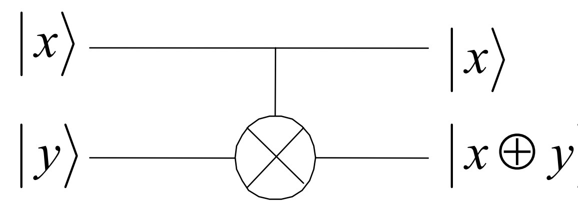
- **By matrix**



Quantum Gate

- **Two-Input Gate:** Controlled NOT (Feynman gate)





- **By matrix**

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 00 \\ 01 \\ 10 \\ 11 \end{pmatrix} = \begin{pmatrix} 00 \\ 01 \\ 11 \\ 10 \end{pmatrix}$$

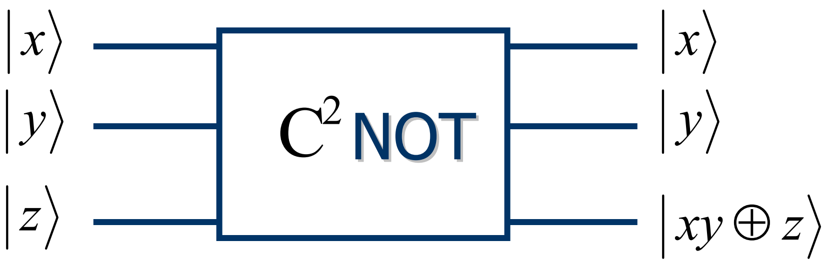
y

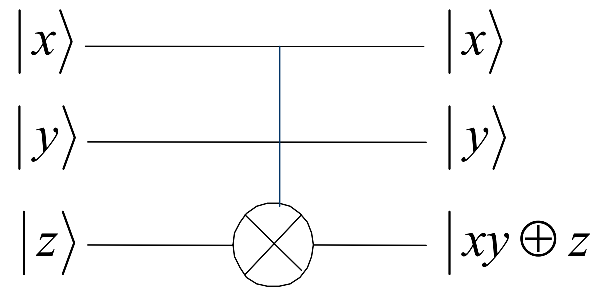
$$|x\rangle|0\rangle \rightarrow |x\rangle|x\rangle$$

$$|x\rangle|1\rangle \rightarrow |x\rangle|\overline{x}\rangle$$

Quantum Gate

- **3-Input gate:** Controlled CNOT (Toffoli gate)

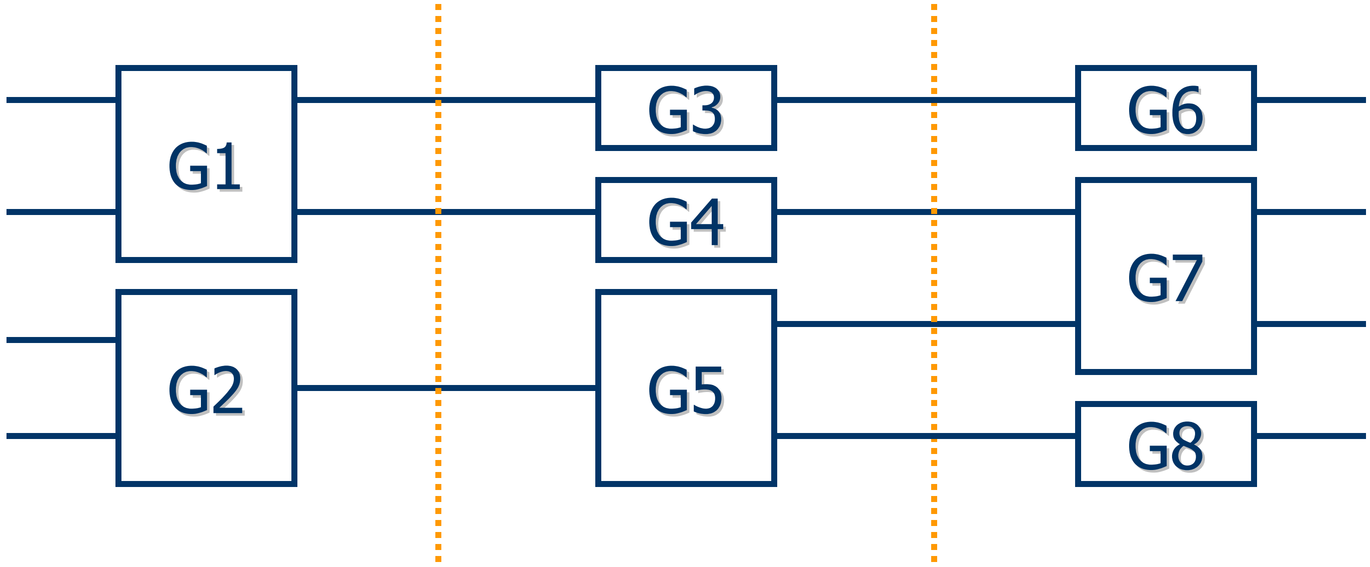




- **By matrix**

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Quantum circuit



Tensor Product → $G1 \otimes G2$ $\quad G3 \otimes G4 \otimes G5$ $\quad G6 \otimes G7 \otimes G8$

Matrix multiplication → $(G6 \otimes G7 \otimes G8) * (G3 \otimes G4 \otimes G5) * (G1 \otimes G2)$