# Additional Perspectives on Fractals*

*Benoit B. Mandelbrot is an IBM Fellow at the IBM Thomas J. Watson Research Center. In one sense he is the discoverer of fractals, the man who found examples of fractals scattered through the work of numerous predecessors in many different fields. In a more important sense he is the inventor of fractals, because the underlying theme that bound together the diverse examples lay unrecognized until he discerned it, named it, and brought it to general attention. The span of Mandelbrot's interests is best illustrated by listing some of the positions he has held: Visiting Professor of Economics, Applied Mathematics, and Mathematics at Harvard; of Engineering at Yale; of Physiology at the Albert Einstein College of Medicine; and of Mathematics at the University of Paris-Sud.*

**Barcellos:** *Now that fractal geometry is, as you say, "taking ominous steps toward becoming organized," at what level of the curriculum would you consider its introduction appropriate?*

**Mandelbrot:** The principal place where fractals should be introduced is with the presentation of the derivative. I have the very strong feeling that many people would understand the derivative better if they are told at the very outset that it does not have to exist. In some cases, a continuous function has no derivative. But now that I have shown that these things are essential in natural science, it's very easy to tell a student that, for example, if you take the motion of a particle along a coastline there is no tangent, there is no derivative. So in this sense I think it is desirable to introduce such notions very early. I have found to my pleasure and surprise that undergraduates understand this stuff very well.

Now yet another place where fractals are doubtless going to be very important is in the teaching of computer methods and computer graphics. I was impressed walking around computer graphics groups in various universities of how often their demonstrations are fractals oriented. There are very few things which are so simple and so effective in the way they are done.

I believe also that a course on topics in fractals would be very useful at a slightly higher level in various departments. For example, the theory of Kleinian groups and of iterations of rational functions are mathematical theories of great complication and great refinement which, in a way, collapsed under their own weight. They became very unmanageable because of the lack of intuitive aspects to them, whereas with fractals these theories are becoming quite easy and attractive. The idea of correspondence between an algorithm and a shape is particularly clearcut in this context.

I am very ambivalent about fractals becoming a field because my work was inspired and supported by a very strong belief that the division of science into fields has been extremely harmful. Nevertheless, I must acknowledge that very often, if

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*An expanded and revised treatment of Mandelbrot's remarks appear in *Mathematical People: Inside the Mathematical Mind*, edited by D. J. Albers and G. L. Alexanderson. This sidebar is reproduced with the kind permission of Birkhäuser Boston, Inc.

one does not maintain a field, then things go in a very bad way. The unity of fractals may have to be maintained at a cost of having a unified course taught somewhere; under some extreme circumstances programs of research in fractals may become appropriate.

**Barcellos:** *The Cantor set and nowhere differentiable functions have long been used as important examples in analysis and topology. Has there been a significant movement toward identifying such ideas as fractals?*

**Mandelbrot:** It's very strange. I go to some places and people speak in terms of fractals. They say it's a fractal and they don't even put the quote marks around it; they even misspell the word, which is a sign that they don't think of it as being a new word. But there is no question that scientists are an extremely conservative group. Therefore, I do know mathematicians who refuse to use the word "fractal" and who continue to speak of my work as being related to generalized Cantor sets or to generalized Peano curves—terms which seem to imply that there is no unity in that endeavor.

**Barcellos:** *They still regard them as pathological then?*

**Mandelbrot:** Well, that's one part of it, but it's not the only part. Take, for example, "strange attractors." The field of strange attractors has arisen and, to my great surprise, the word "strange" did not bother anybody. In fact, they sort of are amused by that, because they turned around a certain term and used it with the contrary of its initial meaning to imply attractors which aren't so strange after all. But the point is that the word "strange" was already part of the vocabulary, and to accept a new word is a step that many people take with reluctance. Mathematicians don't like new words particularly. For example, the vocabulary of mathematics is full of terms like "rings," or "fields," or "complex" numbers and "imaginary" numbers. The word "distribution" must have fifteen different senses.

I also see the word "fractal" used in different meanings from mine. For example, I have some argument with people who try to define fractal dimension in fashions which I find undesirable and which I explained why they were undesirable in my book. The people say, "Well, but we need a slightly different notion and since the word exists we are going to use it." So it may very well be that this word also will not be accepted until it has become ambiguous and slightly less clear than it is now.

**Barcellos:** *Fractals have now been applied in many different fields. Although you were involved in many of these applications from the outset, have there been any new results or applications that have surprised even you?*

**Mandelbrot:** I wouldn't say so. What surprises me constantly is the *quality* of the applications. For example, when I had been saying for a very long time that to describe clouds one must use fractals, Lovejoy started looking at weather very carefully and specifically and found so many aspects for which fractals were the obvious solution that I was surprised. I was expecting this to be more complicated, but in a certain sense things are easier than I thought. So far there is no surprise in the sense of something that really came out of the blue.

**Barcellos:** *Do you have favorites among the fractals, examples that you particularly like?*

**Mandelbrot:** I don't really have favorites because I worked so hard at each of these things for so long that it's like with children—one cannot have favorites really; one likes each for different reasons.

There is one example which is inevitable, just totally unavoidable, which is the coastlines. There is no way of doing it without the coastlines. I think that the example of the vasculature is one which people find quite astonishing but very natural, that to show by discussing the veins and the arteries that not only a surface can have a positive volume but it can have volume bigger than the volume that it surrounds. The example of the mountains has another lesson, that if you try to be too simple you get into trouble. Now I mention that only because of Loren Carpenter. Loren is in a way my disciple, but he tried to simplify the algorithms and to make them easier to run and to program. I find his mountains very bad. It is due to the fact that he tried to simplify it more than one properly can. So if one tries to do so, one destroys the characteristic which is very essential in my mountains—the property of invariance. So the extraordinary thing was that the mountains which are invariant look so good, and the mountains which Carpenter and his friends drew, which are not invariant, look relatively poor.

I think the creation of new shapes, entirely new shapes which nobody has ever seen anywhere, is something which may perhaps be even more important. And that is something which I find now more striking: to look at the screen and see something appearing which really has not existed—not imitation of mountains, not imitation of anything alive, but just new shapes. And they can be so extraordinarily beautiful that it's very rewarding to do it. ■

## MARTIN GARDNER •••

High school students have no difficulty at all understanding something like the snowflake curve and similar variations, and those, of course, are fractal curves. The concept of a fractal curve is well within the grasp of a bright high-school student. It's a very easy subject to introduce on the high-school level, but if you get very much further into the field, then you have to know a little more sophisticated mathematics. So [at the high school level] it would just be an introduction to the concept of the curve.

I first learned about fractals from a Philip Morrison piece reviewing Mandelbrot's book, in which he praised Mandelbrot very highly. Phil Morrison was perceptive enough to see that this was a significant new field of research. I looked Mandelbrot up, and that led to my first column on the subject.

My only contribution to the whole subject of fractals really was introducing Mandelbrot to William Gosper, who had sent me a very interesting fractal curve which he called the "flowsnake"—a takeoff on "snowflake"—and which I published. I had the pleasure of introducing that curve to Mandelbrot, who became intrigued and looked up Gosper. They became very good friends and you'll see a number of references to Gosper's work in Mandelbrot's book. That's about my only contribution—getting Mandelbrot and Gosper together.

I'm just fascinated by the number of applications it has. As you know, it's being used now in creating visual effects for science fiction films. In "Star Trek II," the [fractal] sequence was very short but very impressive.

*Fractal geometry is one of the many beautiful and intriguing topics first introduced to a general readership by its appearance in Martin Gardner's "Mathematical Games" column in Scientific American. Gardner notes that fractals can be treated at vastly different levels of difficulty, providing researchers with unsolved problems and secondary school students with intriguing and unusual examples of geometric objects.* ■

## JAMES W. CANNON •••

Since the end of the last century, fractals have constituted really interesting examples and counter-examples in set theory, topology, and real and complex analysis. These fields have been using fractals for decades; it's simply that he [Mandelbrot] is giving a unified name to these various things.

Now, because of computer graphics, these examples can be visualized in detail even by the unimaginative, so they can probably be expected to appear with even greater prominence than before in undergraduate set theory, topology, and analysis. Fractals can probably come into the curriculum in the early undergraduate career; appropriate places that they might appear would be in a computer graphics course, as a section in a geometry course, or in a section on mathematical modeling.

They're beautiful things that illustrate consequences of simple algorithms. Studying what an algorithm does under iteration is one of the things that students can do and enjoy very early on.

At the research level, fractals currently dominate certain research areas in geometric topology and they will probably continue to do so. They are coming into prominence in the study of Kleinian groups and hyperbolic geometry. There will probably also be special applications as people begin to understand how they can be used as models. One obvious application is to computer art—the most beautiful computer pictures I've seen have been fractals. There will probably be applications to weather with cloud formation, astronomy with matter distribution, geology with land formation, and various others that you've read about and will hear about. In the study of what makes a topological manifold a topological manifold, there are certain difficulties that arise, and trying to understand exactly what those difficulties are leads to a certain fractal which captures exactly the difference between things that are manifolds and things which are just almost manifolds.

Nowhere differentiable functions are fractals. Cantor sets are fractals. Brownian motion is a fractal. "Wild" and "tame" sets [in topology] aren't generally considered fractals by Mandelbrot, but I think they ought to be included. Space-filling curves are fractals.

[Mandelbrot] might not even claim some of these things as fractals, but it seems that in spirit they have the fractal content, and I like to think of them as being unified by that process. Mandelbrot's intuitive definition is very good: It's an object that is irregular or fragmented at all scales. And to make that mathematical, one only needs to choose *any* definition of scale and *any* definition of fragmentation or irregularity and it would fit into the notion of fractal. That's the kind of definition that I use. In topology, the definition of irregularity might be wildness; in geometry, it might be Hausdorff dimension. One could go on and on.

The things that I would choose to call fractals have been around and applicable for a long, long time. As to whether Mandelbrot's physical applications will work or not, I think that hasn't been carried far enough to know.

*Professor James W. Cannon of the University of Wisconsin is a geometric topologist whose research interests include fractals and who has lectured extensively on the subject. His talks on "Topological, Combinatorial and Geometric Fractals," at the MAA Summer Meeting in Toronto, are being prepared for publication in the American Mathematical Monthly.* ■

## LOREN C. CARPENTER •••

I went out and bought the book† as soon as I read Gardner's article. I've gone through it with a magnifying glass two or three times. I found that it was inspirational more than anything else. What I got out of it myself was the notion that "Hey, these things are all over, and if I can find a reasonable mathematical model for making pictures, I can make pictures of all the things fractals are found in." That's why I was quite excited about it.

"Star Trek II" is the only time fractals have ever appeared in a motion picture, in terms of a feature film. The typical frametime for the really rugged mountain scenes averaged an hour and a half on a VAX 11/780. We cannot do that real-time with the machinery that we have and, in fact, I think no one in the world can do it with the machinery anybody has. There's just too much work.

The method I use is recursive subdivision, and it has a lot of advantages for the applications that we're dealing with here; that is, extreme perspective, dynamic motion, local control—if I want to put a house over here, I can do it. The subdivision process involves a recursive breaking-up of large triangles into smaller triangles. We can adjust the fineness of the precision that we use. For example, in "Star Trek," the images were not computed to as fine a resolution as possible because it's an animated sequence and things are going by quickly. You can see little triangles if you look carefully, but most people never saw them.

Mandelbrot and others who have studied these sorts of processes mathematically have long been aware that there are recursive approximations to them, but the idea of actually using recursive approximation to make pictures, a computer graphics-type application, as far as we know first occurred to myself and two fellows, Alain Fournier and Don Fussell, in 1979. We each took off on a different track—I went with the triangular subdivision, and they went with modified normals and patches [Fournier].

One of the major problems with fractals in synthetic imagery is the control problem. They tend to get out of hand. They'll go random all over the place on you. If you want to keep a good tight fist on it and make it look like what you want it to look like, it requires quite a bit of tinkering and experience to get it right. There aren't many people around who know how to do it.

*Loren C. Carpenter is a member of the Graphics Project team at Lucasfilm Limited in San Rafael, California, the production company of George Lucas, creator of "Star Wars." His application of fractals to computer graphics gave us the sequence of synthetic images used to depict the transformation of a planet in the motion picture "Star Trek II: The Wrath of Khan." He was introduced to the first English edition of Mandelbrot's book on fractals by Gardner's original column on the subject in Scientific American.* ■

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†*Editor's Note:* Benoit Mandelbrot's book *The Fractal Geometry of Nature* is reviewed in this issue, pp. 178–180.