Branch and bound algorithm for covering
Reduction strategies

- Partitioning:
  - If $A$ is block diagonal:
    - Solve covering problem for corresponding blocks.

- Essentials:
  - Column incident to one (or more) row with single 1:
    - Select column.
    - Remove covered row(s) from table.

Discuss the historic example of essential subset and function core
Branch and bound algorithm for covering. Reduction strategies

- **Column dominance:**
  - If $a_{ki} \geq a_{kj} \ \forall k$:
    - remove column $j$.

- **Row dominance:**
  - If $a_{ik} \geq a_{jk} \ \forall k$:
    - Remove row $i$. 
Example

\[
A = \begin{bmatrix}
1 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 \\
\end{bmatrix}
\]
Example reduction

- Fourth column is essential.
- Fifth column is dominated.
- Fifth row is dominant.

\[
A = \begin{pmatrix} 
1 & 0 & 1 \\
1 & 1 & 0 \\
1 & 1 & 0 \\
0 & 1 & 1 \\
\end{pmatrix}
\]
EXACT COVER( A; x; b) f
Reduce matrix A and update corresponding x;
if (Current estimate j bj ) return(b);
if ( A has no rows ) return (x);
Select a branching column c;
xc =1 ;
e
A = A after deleting c and rows incident to it;
e
x =EXACT COVER( 
e
A; x; b);
if ( j 
e
xj < j bj )
b =
e
x ;
xc =0 ;
e
A = A after deleting c ;
e
x =EXACT COVER( 
e
A; x; b);
if ( j 
e
Bounding function

- Estimate lower bound on the covers derived from the current $x$.
- The sum of the ones in $x$, plus bound on cover for local $A$:
  - Independent set of rows:
    - No 1 in same column.
  - Build graph denoting pair-wise independence.
  - Find clique number.
  - Approximation by defect is acceptable.
Example

\[ A = \begin{pmatrix}
1 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 
\end{pmatrix} \]

- Row 4 independent from 1, 2, 3.
- Clique number is 2.
- Bound is 2.
Example

\[ A = \begin{pmatrix} 
1 & 0 & 1 \\
1 & 1 & 0 \\
0 & 1 & 1 
\end{pmatrix} \]

- There are no independent rows.
- Clique number is 1 (one vertex).
- Bound is 1 + 1 (already selected essential).
Example

\[
A = \begin{pmatrix}
  1 & 0 & 1 \\
  1 & 1 & 0 \\
  0 & 1 & 1 \\
\end{pmatrix}
\]

- Choose first column:
  - Recur with \( \bar{A} = [11] \).
    - Delete one dominated column.
    - Take other column (essential).
  - New cost is 3.

- Exclude first column:
  - Find another solution with cost 3 (discarded).
Unate and binate cover

• Set covering problem:
  – Involves a unate clause.

• Covering with implications:
  – Involves a binate clause.

• Example:
  – The choice of an element implies the choice of another element.
Unate and binate covering problems

- **Unate cover:**
  - Exact minimization of Boolean functions.

- **Binate cover:**
  - Exact minimization of Boolean relations.
  - Exact library binding.
  - Exact state minimization.
Unate and binate covering problems

• **Unate cover:**
  – It always has a solution.
  – Adding and element to a feasible solution preserves feasibility.

• **Binate cover:**
  – It may not have a solution.
  – *Adding and element to a feasible solution may make it unfeasible.*
  – Minimum-cost satisfiability problem.
  – Intrinsically more difficult.
Algorithms for unate and binate covering

- Branch and bound algorithm:
  - Extended to weighted covers.
- More complex in the binate case:
  - Dominant clauses can be discarded only if weight dominates.
  - Harder to bound.
- Only problems of smaller size are solvable, comparing to unate.
- Heuristic for binate cover are also more difficult to develop.