A Fast Heuristic Algorithm for Disjoint Decomposition of Boolean Functions

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Abstract

This paper presents a heuristic algorithm for disjoint decomposition of a Boolean function based on its ROBDD representation. Two distinct features make the algorithm feasible for large functions. First, for an n-variable function, it checks only $O(n^2)$ candidates for decomposition out of $O(2^n)$ possible ones. A special strategy for selecting candidates makes it likely that all other decompositions are encoded in the selected ones. Second, the decompositions for the approved candidates are computed using a novel IntervalCut algorithm. This algorithm does not require reordering of ROBDD. The combination of both techniques allows us to decompose the functions of size beyond that possible with the exact algorithms. The experimental results on 582 benchmark functions show that the presented heuristic finds 95% of all decompositions on average. For 526 of those functions, it finds 100% of the decompositions.

1 Introduction

The *disjoint decomposition* of a Boolean function is a representation of type $f(X) = h(g(Y),Z)$ with Y and Z being sets of variables partitioning the set X. Disjoint decomposition has many applications in computer science and discrete mathematics, including logic synthesis (decomposition of Boolean functions), reliability theory (decomposition of coherent systems [1]), game theory (decomposition of simple *n*-persons games [2]) and combinatorial optimization problems over graphs and networks (see [3] for an overview).

This wide range of applications makes it important to have efficient algorithms for finding all, or at least some, decompositions for a given structure. Fast decomposition algorithms are known for binary relations and graphs [4, 5, 6]. For Boolean functions, however, the existing methods either involve the solution of an NP-complete problem (as in [7]) or have exponential running time [8, 9, 10, 11]. More recent ROBDD-based decomposition algorithms, including [12, 13, 14], show much better average-time performance.

This paper presents a heuristic algorithm targeting to find all disjoint decompositions of an *n*-variable Boolean function represented by a ROBDD. The heuristic is based on two properties: (1) all decompositions of a Boolean function (which can be $O(2^n)$) can be uniquely described by a certain subset of decompositions A (which is only $O(n)$); (2) there exist a best variable ordering for a ROBDD in which the variables Y from any decomposition $f(X) = h(g(Y),Z)$ belonging to A are adjacent.

If we had such a best ordering, we could examine all its linear intervals to find which Y results in a decomposition $f(X) = h(g(Y),Z)$. However, computing best orderings is infeasible for large functions. The algorithm presented in this paper is *heuristic* because it starts from a "good" ordering which is not necessarily keeping the variables Y adjacent. The experimental results show that if *sifting* ordering algorithm [15] is used to get a "good" initial order, then our heuristic finds 95% of all decompositions on average. The presented heuristic algorithm is also able to decompose functions which are too large for the exact algorithms.

2 Previous work

The first major investigation on the subject was carried out by Ashenhurst [16]. He studied simple disjoint decomposition $f(X) = h(g(Y),Z)$ for Boolean functions $f,g,h : B^n \to B$, where $B = \{0,1\}$. Ashenhurst's fundamental contribution is a theorem which states that any Boolean function has a unique *disjoint tree-like decomposition* such that all possible simple disjoint decompositions of f are exhibited.

Curtis [17] and Roth and Karp [18] extended Ashenhurst theory to the decomposition of type $f(X) = h(g(Y),Z)$ with g,H being multiple-valued functions of type $g: B^{|Y|} \to M$ and $h: M \times B^{|Z|} \to B$, where $M = \{0,1,\ldots,m-1\}$. The function g can be encoded by $k = \lceil \log_2 m \rceil$ Boolean functions $g_1, g_2, \ldots, g_k$, giving a decomposition of the form $f(X) = h(g_1(Y),\ldots,g_k(Y),Z)$, often referred to as *Roth-Karp* decomposition. Unfortunately Ashenhurst's main theorem does not extend directly to multiple-valued functions (for a counterexample see chapter 4 of [19]). A consequence of this is that there is no unique disjoint tree-like Roth-Karp decomposition. Von Stengel [20] has defined a class of multiple-valued functions for which Ashenhurst's main theorem holds.

Early algorithms for decomposition used *decomposition charts* [16], [17]. The decomposition chart for $f(Y,Z)$ is a two-dimensional table where the columns represent all combinations of the variables from the set Y and the rows represent all combinations of the variables from the set Z. The set Y is a bound set if and only if the chart has *column multiplicity* at most two, i.e. there are at most two distinct columns in the chart [16].

In a short time, decomposition charts were abandoned in favor of *cube* representation [21]. The task of computing column multiplicity on charts was replaced by the task of computing *compatible classes* for a set of cubes. Two assignments $x_1, x_2 \in B^{|Y|}$

are said to be *compatible with respect to the reference function* $f(Y,Z)$ if, for all $y \in B^{|Z|}$ such that $f(x_1,y)$ and $f(x_2,y)$ are defined, $f(x_1,y) = f(x_2,y)$ [21]. The set Y is a bound set if and only if $B^{|Y|}$ can be partitioned into $k \leq 2$ mutually compatible classes [21]. If $f(X)$ is completely specified, then compatibility is an equivalence relation and k is the number of equivalence classes. It is easy to see the one-to-one mapping between a column in a decomposition chart and a compatible class.

Due to the exponential size of decomposition charts and cube representations, early decomposition algorithms were rarely applied to large practical circuits. Instead, *algebraic* methods were used [22]. ROBDDs [23] made possible developing new algorithms for decomposition, feasible for much larger functions than previously possible.

In a ROBDD, the column multiplicity can be easily computed by moving the variables Y to the upper part of the graph and checking the number of children below the boundary line, usually called *cut* line. The decomposition $f(X) = h(g(Y),Z)$ exists if and only if there are only two children below the cut line [24].

This approach has been adopted by a number of BDD-based decomposition algorithms [24, 25, 26, 27]. Stanion and Sechen [28] used cut to find *quasi-algebraic* decomposition of the form $f(X) = g(Y) \odot h(Z)$, where "$\odot$" is any binary Boolean operation and $|Y \cup Z| = k$ for some $k \geq 0$. This type decomposition is often referred to as *bi-decomposition* [29, 30].

BDD-based decomposition algorithms following cut-strategy proved to be orders of magnitude faster than those based on decomposition charts and cube representations. However, they require reordering of variables of BDD to move the variables on the top or to check bi-decompositions for partitionings which are not consistent with the variable order. As an alternative, a number of methods use the fact that BDDs themselves are a decomposed representation of the function and exploit the structure of BDDs, rather than cut, to find disjoint decompositions. Karplus [31] extended the classical concept of *dominator* on graphs [32] to 0,1-dominators on BDDs. A node v is a 1-dominator (0-dominator) if every path from the root to one (zero) terminal node contains v. If v is a 1-dominator, then the function represented by the BDD possesses a conjunctive (AND) decomposition. If v is a 0-dominator, then the function can be decomposed disjunctively (OR). This idea was extended by Yang et al [33] to XOR-type decompositions and to more general type of dominators. Minato and De Micheli [13] presented an algorithm which computes disjoint decompositions by generating irreducible sum-of-product for the function from its BDD and applying factorization. The algorithm of Bertacco and Damiani [12] makes a single traversal of the BDD to identify the decomposition of the co-factors and then combine them to obtain the decomposition for the entire function. The algorithm is impressively fast; however, as Sasao has observed in [34], it fails to compute some of the disjoint decompositions. This problem was corrected by Matsunaga [14], who added the missing cases in [12] allowing to treat the OR/XOR functions correctly. The algorithm [14] appears to be the fastest of existing exact algorithms for finding all disjoint decompositions.

3 New heuristic algorithm

The new heuristic algorithm is based on the following two properties.

Property 1 *All disjoint decompositions of an n-variable Boolean function can be uniquely described by a certain subset of disjoint decompositions A. The size of A is* $O(n)$.

Property 2 *There exist a best variable ordering for a ROBDD for* f *in which the variables* Y *from any decomposition* $f(X) = h(g(Y),Z)$ *belonging to A are adjacent.*

Property 1 follows from the results of [20]. We describe these results briefly in Section 3.1. Property 2 follows from the main theorem of [35].

The presented algorithm examines all linear intervals of variables from a given ordering of a ROBDD and, for each interval Y, checks whether it is a bound set. The procedure **IntervalCut** described in Section 3.2, is used to perform the checking as well as to compute the functions g and h in the resulting decomposition $f(X) = h(g(Y),Z)$.

3.1 Properties of the disjoint decomposition

This section describes the properties of the disjoint decomposition from [20], implying Property 1. The formulation of the definitions and theorems is adjusted to the notation of this paper.

Definition 1 *A bound set* Y *of* $f(X)$, $Y \subset X$, *is strong if any other bound set of* $f(X)$ *is either a subset of* Y, *a superset of* Y, *or disjoint to* Y.

The partial order induced by set theoretical inclusion between pairs of strong bound sets of f defines a tree.

Definition 2 *The decomposition tree* $T(f)$ *of* $f(X)$ *is a tree whose nodes represent all strong bound sets of* $f(X)$, *related by inclusion. Any node has two labels:*
(a) a type, which is either "prime" or "full",
(b) an associated function.

The following Theorem shows how decompositions of a function can be derived from its decomposition tree and characterizes the functions associated with the nodes. It also states that the decomposition tree is unique for a given function (up to isotopy/isomorphy). Remind that two Boolean functions are *isotopic* if they are identical up to complementation of variables or function values. Two binary operations $\circ$ and $\bullet$ are *isomorphic* if there is a bijection $\phi : B \to B$ such that $\phi(a \circ b) = \phi(a) \bullet \phi(b)$.

Theorem 1 *Let* $T(f)$ *be the decomposition tree of a Boolean function* $f(X)$ *with support set* X. *Let* $Y_1, \ldots, Y_k$ *be the children of the root* X. *Then* $f(X)$ *has a decomposition of type*

$$f(X) = h(g_1(Y_1), g_2(Y_2), \ldots, g_k(Y_k))$$

for functions $g_i : B^{|Y_i|} \to B$ $(1 \leq i \leq k)$ *and* $h : B^k \to B$ *where*
(a) h is non-decomposable if X is labeled "prime",
(b) h is an associative and commutative Boolean operation if X is labeled "full",
(c) h is unique up to isotopy in (a) and up to isomorphy in (b).

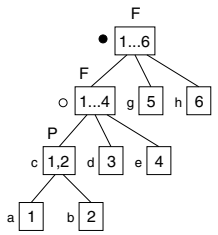

Figure 1: Example of a decomposition tree.

An example of a decomposition tree is shown in Figure 1. Abbreviations "P" and "F" stand for labels "prime", and "full", respectively. Letters a,b,c,d,e,g,h denote the functions associated with the nodes, whereas $\bullet$ and $\circ$ denote operations. In accordance with the tree, the complete disjoint decomposition of the function is

$$f(x_1,\ldots,x_6) = (c(a(x_1),b(x_2)) \circ d(x_3) \circ e(x_4)) \bullet g(x_5) \bullet h(x_6)$$

with $\bullet$ and $\circ$ being associative and commutative Boolean operations. a,b,c,d,e,g,h are non-decomposable Boolean functions. In this case all those functions except c are unary Boolean functions (identity or complement).

Theorem 1 shows that the decompositions associated with strong bound sets uniquely represent all disjoint decompositions of a function. These are the decompositions A of Property 1. It was proved in [36] that the number of strong bound sets of an n-variable Boolean function is $O(n)$, while the number of all bound sets is $O(2^n)$.

3.2 IntervalCut procedure for finding bound sets

Let V be a set of nodes of a ROBDD G of an n-variable function $f(X)$. Every non-terminal node $v \in V$ has an associated variable index, $index(v) \in \{1,\ldots,n\}$. The index of the root node is 1. In order to have a unified notation in the proof of the main result, we assume that the terminal nodes also have an index, which is $n+1$.

Suppose that all nodes with $index \leq i$ are in the upper part of the graph and all nodes with $index > i$ are in the lower part of the graph, for some $i \in \{1,\ldots,n\}$. The boundary line between the upper and lower parts of the graph is called $cut(i)$. If the number of nodes with $index > i$ which are children of the nodes above the $cut(i)$ is two, then the set of variables $Y = \{x_1,\ldots,x_i\}$ is a bound set [25].

One possibility to check whether a set of variables Y is a bound set is to move the variables Y to the top of the ROBDD and then check the number of children below $cut(|Y|)$, as in [25, 26]. However, re-ordering is computationally expensive. Instead, we have developed a procedure, called **Interval Cut** which checks whether a given linear interval of variables of a ROBDD is a bound set without reordering. To describe the procedure, we first introduce some definitions.

Suppose the variables Y lie between two cuts, $cut(a)$ and $cut(b)$, such that $a < b$, $a,b \in \{0,\ldots,n\}$. Let $cut\_set(a)$ denote a set of nodes $v \in G$ with indexes $a < index(v) \leq b$ which are children of the nodes above the $cut(a)$ of G. Let G_v stand for a ROBDD rooted at some $v \in cut\_set(a)$. Then, $cut\_set(b_v)$ is the set of nodes $u \in G_v$ with indexes $b < index(u) \leq n+1$ which are children of the nodes of G_v above the $cut(b)$. If $|cut\_set(b_v)| = 2$, then g_v is a Boolean function represented by the subgraph rooted at v whose terminal nodes are obtained by replacing the two nodes of $cut\_set(b_v)$. The resulting g_v is unique up to complementation.

IntervalCut(G,a,b)
input: ROBDD G of $f(X)$, two cuts $cut(a)$ and $cut(b)$, $a < b$, $a,b \in \{0,\ldots,n\}$.
output: "not a bound set" if the set of variables Y between $cut(a)$ and $cut(b)$ is not a bound set of $f(X)$; functions g and h if Y is a bound set resulting in $f(X) = h(g(Y),Z)$.

for *all* $v \in cut\_set(a)$
  if $(|cut\_set(b_v)| > 2)$
    return(*"not a bound set"*);
for *all* $v_1, v_2, \ldots, v_k \in cut\_set(a)$
  if $(g_{v_i} \neq g_{v_{i+1}})$ /* *up to complementation* */
    return(*"not a bound set"*);
h = substitute each subgraph g_v, $\forall v \in cut\_set(a)$, *by a node;*
$g = g_v$;
return(g,h);

Figure 2: Pseudo code of the **IntervalCut** procedure.

Using this notation, we can describe the pseudo code of the algorithm **IntervalCut**(G,a,b) as shown in Figure 2. Next, we prove that it computes the decompositions correctly.

Theorem 2 *Algorithm* **IntervalCut**(G,a,b) *computes a decomposition* $f(X) = h(g(Y),Z)$ *in* $O(|cut\_set(a)| \cdot max(|g_v|))$ *time,* $v \in cut\_set(a)$.

Proof: Let Y be the variables between $cut(a)$ and $cut(b)$, Z_1 be the variables above $cut(a)$ and Z_2 be the variables below $cut(b)$. We have $Z_1 \cup Z_2 = Z$ and $Y \cup Z = X$.

Let $k_v(Z_1)$ be a function which is a sum of all the paths leading to a node $v \in cut\_set(a)$. Then f can be co-factored with respect to k_v as

$$f(X) = \sum_{\forall v \in cut\_set(a)} k_v(Z_1) \cdot f|_{k_v}(Y,Z_2) \qquad (1)$$

If $|cut\_set(b_v)| = 2$, then Y is a bound set for $f|_{k_v}$ so it can be decomposed as

$$f|_{k_v}(Y,Z_2) = h_v(g_v(Y),Z_2) \qquad (2)$$

for some h_v, g_v. Furthermore, if for all $v \in cut\_set(a)$ the functions g_v are equal up to complementation, then we can denote g_v by g and write (2) as

$$f|_{k_v}(Y,Z_2) = h_v(g(Y),Z_2) \qquad (3)$$

From (1) and (3) we can conclude that f can be represented as

$$f(X) = h(g(Y),Z)$$

with $h = \sum_{\forall v \in cut\_set(a)} k_v \cdot h_v$.

Let $max(|g_v|)$ be the size of the largest subgraph representing g_v, for some $v \in cut\_set(a)$. Since substitution of a ROBDD by a node is a constant-time operation, the complexity of the pseudo code in Figure 2 is $O(|cut\_set(a)| \cdot max(|g_v|))$.

□

4 Experimental results

To make a thorough evaluation of the presented heuristic, we have implemented an exact decomposition algorithm[1] from [37] and applied both, exact and heuristic versions, to *iwls*93 benchmark set. For all single outputs, for which the exact algorithm did not time out[2], 582 in total, we have computed the total number of strong bound sets found by each algorithm. In the first set of experiments, we used *sifting* ordering algorithm [15] to get a good initial order for ROBDDs. The heuristic algorithm has succeeded to find 95% of all the decompositions on average. For 526 of those 582 single-output functions, it found 100% of the decompositions. In the second set of experiments, we switched the sifting off, and build ROBDDs using the breadth first traversal order from the benchmark's circuit description. For 191 functions out of 582 the result got worse (by 57% on average). Nevertheless, the heuristic still found all the decompositions for 365 functions.

We have also applied the presented heuristic to the benchmarks reported in [13], [12] and [14]. The results are summarized in Table 1. Column 4 shows how many non-trivial strong bound sets are found for each benchmark by our algorithm. Every output is handled as a separate function. The number given in Column 4 is the total sum of bound sets for all the outputs. Columns 5-8 show runtime comparison. Our experiments were run on Sun Ultra 60 operating with two 360 MHz CPU and with 1024 MB RAM main storage. The algorithm [13] uses a SUN Ultra 30, [12] uses a PC equipped with 150 MHz Pentium and 96 MB RAM main storage and [14] uses a PC with Pentium-II 233Mhz processor.

name	in	out	N of bound sets	CPU time, sec: presented heuristic	exact alg [13]	exact alg [12]	exact alg [14]
alu2	10	6	3	0.0002	-	0.28	-
alu4	14	8	2	0.0009	-	0.37	0.15
apex1	45	45	83	0.008	59.0	1.01	-
apex2	38	3	16	0.001	5.9	1.14	-
apex3	54	50	23	0.008	44.3	-	-
apex4	9	19	4	0.002	-	0.33	-
apex5	114	88	196	0.032	-	2.34	-
apex6	135	99	258	0.008	13.1	2.62	0.41
apex7	49	37	96	0.006	1.7	1.03	0.37
b9	41	21	49	0.001	-	-	0.02
C432	36	7	10	0.002	415.4	1.23	0.28
C499	41	32	68	5.2	-	83.47	8.80
C880	60	26	45	0.046	-	2.71	0.92
C1355	41	32	0	5.2	-	91.25	8.87
C1908	33	25	15	0.23	-	7.58	1.42
C3540	50	22	18	2.8	-	21.1	3.48
cmb	16	4	4	0.002	-	0.36	-
CM42	4	10	10	0.0006	-	0.15	-
CM85	11	3	15	0.0003	-	0.27	-
CM150	21	1	1	<0.0001	-	0.51	-
comp	32	3	47	0.002	-	0.71	-
count	35	16	47	0.007	-	0.73	0.01
dalu	75	16	42	0.015	>0.8	-	-
des	256	245	688	0.041	-	-	0.36
e64	65	65	63	0.51	-	1.31	-
f51m	8	8	6	0.0004	-	0.26	-
frg2	143	139	532	0.032	19.2	2.86	0.15
k2	45	45	85	0.008	-	1.04	-
lal	26	19	57	0.002	-	0.55	-
misex2	25	18	29	0.003	-	0.57	-
mux	21	1	1	0.0001	-	0.48	-
pair	173	137	725	0.040	-	4.02	7.36
PARITY	16	1	1	0.001	-	0.38	-
rot	135	107	296	0.039	-	22.62	-
seq	41	35	135	0.009	67.8	1.10	-
s298	17	20	15	0.0004	-	0.40	-
s420	35	18	18	0.007	-	0.75	-
s444	24	27	65	0.001	-	0.54	-
s526	24	27	45	0.002	-	0.52	-
s641	54	42	138	0.003	-	1.12	-
s832	23	24	37	0.003	-	0.54	-
s953	45	52	40	0.003	-	20.97	-
s1196	32	32	33	0.002	-	0.71	-
s1238	32	32	33	0.002	-	0.75	-
s1423	91	79	38	0.066	-	12.48	-
s1488	14	25	38	0.002	-	0.36	-
s1494	14	25	38	0.002	-	0.34	-
term1	34	10	65	0.002	-	0.75	-
too_large	38	3	17	0.001	>1.0	-	0.09
ttt2	24	21	44	0.002	-	0.55	-
vda	39	17	30	0.003	>0.5	0.4	-
x3	135	99	278	0.008	-	2.69	-
x4	94	71	180	0.008	-	1.90	-

Table 1: Experimental results; "$-$" indicates that information for the benchmark is not provided; "$>$" indicates that information is only provided for one of the outputs.

5 Conclusion

This paper presents a heuristic algorithm for finding disjoint decompositions of Boolean functions. Benchmark experiments demonstrate the effectiveness of the described technique.

Future work includes extension of the presented algorithm to Roth-Karp decomposition. We are also investigating a possibility of combining **IntervalCut** with decomposition algorithms exploiting the structure of BDDs, like [14].

Acknowledgment

This work was supported in part by IBM Partnership Award.

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[1] We have chosen [37] because this algorithm actually builds decomposition trees. It computes only $O(n)$ strong bound sets which are the nodes of $T(f)$.

[2] Time limit 30 min per circuit.

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