Some observations on two-way finite automata with quantum and classical states

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Abstract

Two-way finite automata with quantum and classical states (2qcfa's) were introduced by Ambainis and Watrous. Though this computing model is more restricted than the usual *two-way quantum finite automata* (2qfa's) first proposed by Kondacs and Watrous, it is still more powerful than the classical counterpart. In this note, we focus on dealing with the operation properties of 2qcfa's. We prove that the Boolean operations (intersection, union, and complement) and the reversal operation of the class of languages recognized by 2qcfa's with error probabilities are closed; as well, we verify that the catenation operation of such class of languages is closed under certain restricted condition. The numbers of states of these 2qcfa's for the above operations are presented. Some examples are included, and $\{xx^R | x \in \{a,b\}^*, \#_x(a) = \#_x(b)\}$ is shown to be recognized by 2qcfa with one-sided error probability, where x^R is the reversal of x, and $\#_x(a)$ denotes the a's number in string x.

Keywords: Quantum finite automata; operations; quantum computing.

1. Introduction

Quantum computers—the physical devices complying with quantum mechanics were first suggested by Feynman [15] and then formalized further by Deutch [12]. A main goal for exploring this kind of model of computation is to clarify whether computing models built on quantum physics can surpass classical ones in essence. Actually, in 1990's Shor's quantum algorithm for factoring integers in polynomial time [30] and afterwards Grover's algorithm of searching in database of size n with only $O(\sqrt{n})$ accesses [17] have successfully shown the great power of quantum computers. Since then great attention has been given to this intriguing field in the academic community [19,27], in which the study of clarifying the power of some fundamental models of quantum computation is of interest [19, pp. 151-192].

Quantum finite automata (qfa's) can be thought of theoretical models of quantum computers with finite memory. With the rise of exploring quantum computers, this kind of theoretical models was firstly studied by Moore and Crutchfield [24], Kondacs and Watrous [23], and then Ambainis and Freilds [1], Brodsky and Pippenger [11], and the other authors (e.g., name only a few, [2,4,5,7,8,9,10,18,25,26,28,29], and for some details we may refer to [19]). The study of qfa's is mainly divided into two ways: one is *one-way quantum finite automata* (1qfa's) whose tape heads move one cell only to right at each evolution, and the other *two-way quantum finite automata* (2qfa's), in which the tape heads are allowed to move towards right or left, or to be stationary. (Notably, Amano and Iwama [3] dealt with 1.5qfa's whose tape heads are allowed to move right or to be stationary, and showed that the emptiness problem for this restricted model is undecidable.) In terms of the measurement times in a computation, 1qfa's have two types: *measure-once* 1qfa's (MO-1qfa's) initiated by Moore and Crutchfield [24] and *measure-many* 1qfa's (MM-1qfa's) studied firstly by Kondacs and Watrous [23].

MO-1qfa's mean that at every computation there is only a measurement at the end of computation, whereas MM-1qfa's represent that measurement is performed at each evolution. The class of languages recognized by MM-1qfa's with bounded error probabilities strictly bigger than that by MO-1qfa's, but both MO-1qfa's and MM-1qfa's recognize proper subclass of regular languages with bounded error probabilities [1,24,11,23,7,8]. On the other hand, the class of languages recognized by MM-1qfa's with bounded error probabilities is not closed under the binary Boolean operations (intersection, union, complement) [1,4,11,8], and by contrast MO-1qfa's satisfy the closure properties of the languages recognized with bounded error probabilities under binary Boolean operations [11,10].

A more powerful model of quantum computation than its classical counterpart is 2qfa's that were first studied by Kondacs and Watrous [23]. As is well known, classical two-way finite automata have the same power as one-way finite automata for recognizing languages. Freivalds [16] proved that *two-way probabilistic finite automata* (2pfa's) can recognize non-regular language $L_{eq} = \{a^n b^n | n \in \mathbf{N}\}$ with arbitrarily small error, but it was verified to require exponential expected time [20]. (In this paper, $\mathbf{N}$ denotes the set of natural numbers.) Furthermore, it was demonstrated that any 2pfa's recognizing non-regular languages with bounded error probabilities need take exponential expected time [13,22]. In 2qfa's, a sharp contrast has arisen, as Kondacs and Watrous [23] proved that L_{eq} can be recognized by some 2qfa's with one-sided error probability in linear time.

Recently, Ambainis and Watrous [6] proposed a different two-way quantum computing model—*two-way finite automata with quantum and classical states* (2qcfa's). In this model, there are both quantum states and classical states, and correspondingly two transfer functions: one specifies unitary operator or measurement for the evolution of quantum states and the other describes the evolution of classical part of the machine, including the classical internal states and the tape head. Therefore, this model can be viewed as an intermediate version between 1qfa's and 2qfa's, and it is more restricted than ordinary 2qfa's by Kondacs and Watrous [23]. This device may be simpler to implement than ordinary 2qfa's, since the moves of tape heads of 2qcfa's are classical. In spite of the existing restriction, 2qcfa's have more power than 2pfa's. Indeed, as Ambainis and Watrous [6] pointed out, 2qcfa's clearly can recognize all regular languages with certainty, and particularly, they [6] proved that this model can also recognize non-regular languages $L_{eq} = \{a^n b^n | n \geq 1\}$ and palindromes $L_{pal} = \{x \in \{a, b\}^* | x = x^R\}$, where notably the complexity for recognizing L_{eq} is polynomial time in one-sided error. As is known, no 2pfa can recognize L_{pal} with bounded error in any amount of time [14]. Therefore, this is an interesting and more practicable model of quantum computation, and we hope to deal with further related basic properties.

Operations of finite automata are of importance [21] and also interest in the framework of quantum computing. Our goal in this note is to deal with the operation properties of 2qcfa's. We investigate some closure properties of the class of languages recognized by 2qcfa's, and we focus on the binary Boolean operations, reversal operation, and catenation operation. Notwithstanding, we do not know whether or not these properties hold for the ordinary 2qfa's without any restricted condition, and would like to propose them as an open problem (As the author is aware, the main problem to be overcome is how to preserve the unitarity of the constructed 2qfa's without any restricted condition).

The remainder of the paper is organized as follows. In Section 2 we introduce the definition of 2qcfa's and related results; as well, in terms of the results by Ambainis and Watrous [6], we further present some non-regular languages recognized by 2qcfa's with one-sided error probabilities in polynomial expected time. Section 3 is the main part and deals with operation properties of 2qcfa's, including intersection, union, complement, reversal, and catenation operations; also, we include some examples as an application of these results derived, and we present the numbers of states of these 2qcfa's for the above operations. Finally, some remarks are included in Section 4.

2. Definition of 2qcfa's and some non-regular languages related

In this section, we recall the definition of 2qcfa's, and, introduce the 2qcfa for accepting L_{eq} with one-sided error probability in polynomial time that was verified by Ambainis and Watrous [6].

A 2qcfa M consists of a 9-tuple

$$M = (Q, S, \Sigma, \Theta, \delta, q_0, s_0, S_{acc}, S_{rej})$$

where Q and S are finite state sets, representing quantum states and classical states, respectively, Σ is a finite alphabet of input, $q_0 \in Q$ and $s_0 \in S$ denote respectively the initial quantum state and classical state, $S_{acc}, S_{rej} \subseteq S$ represent the sets of accepting and rejecting, respectively, Θ and δ are the functions specifying the behavior of M regarding quantum portion and classical portion of the internal states, respectively.

For describing Θ and δ, we further introduce related notions. We denote $\Gamma = \Sigma \cup \{\text{¢}, \$\}$, where ¢ and \$ are respectively the left end-marker and right end-marker. $l_2(Q)$ represents the Hilbert space with the corresponding base identified with set Q. Let $\mathcal{U}(l_2(Q))$ and $\mathcal{M}(l_2(Q))$ denote the sets of unitary operators and orthogonal measurements over $l_2(Q)$, respectively. An orthogonal measurement over $l_2(Q)$ is described by a finite set $\{P_j\}$ of projection operators on $l_2(Q)$ such that $\sum_j P_j = I$ and $P_iP_j = \begin{cases} P_j, & i = j, \\ O, & i \neq j, \end{cases}$ where I and O are identity operator and zero operator on $l_2(Q)$, respectively. If a superposition state $|\psi\rangle$ is measured by an orthogonal measurement described by set $\{P_j\}$, then

1. the result of the measurement is j with probability $\|P_j|\psi\rangle\|^2$ for each j,
2. and the superposition of the system collapses to $P_j|\psi\rangle/\|P_j|\psi\rangle\|$ in case j is the result of measurement.

For example, suppose $Q = \cup_j Q_j$ and $Q_i \cap Q_j = \emptyset$ for any $i \neq j$, then all the projectors P_j mapping to subspaces *span* Q_j spanned by Q_j specify an orthogonal measurement over $l_2(Q)$.

Θ and δ are specified as follows. Θ is a mapping from $S\backslash(S_{acc} \cup S_{rej}) \times \Gamma$ to $\mathcal{U}(l_2(Q)) \cup \mathcal{M}(l_2(Q))$, and δ is a mapping from $S\backslash(S_{acc} \cup S_{rej}) \times \Gamma$ to $S \times \{-1, 0, 1\}$. To be more precise, for any pair $(s, \sigma) \in S\backslash(S_{acc} \cup S_{rej}) \times \Gamma$,

1. if $\Theta(s, \sigma)$ is a unitary operator U, then U performing the current superposition of quantum states evolves into new superposition, and $\delta(s, \sigma) = (s^{'}, d) \in S \times \{-1, 0, 1\}$ makes the current classical state s become $s^{'}$, together with the tape head moving in terms of d (moving right one cell if $d = 1$, left if $d = -1$, and being stationary if $d = 0$), for which in case $s^{'} \in S_{acc}$, the input is accepted, and in case $s^{'} \in Q_{rej}$, the input rejected;
2. if $\Theta(s, \sigma)$ is an orthogonal measurement, then the current quantum state, say $|\psi\rangle$, is naturally changed to quantum state $P_j|\psi\rangle/\|P_j|\psi\rangle\|$ with probability $\|P_j|\psi\rangle\|^2$ in terms of the measurement, and in this case, $\delta(s, \sigma)$ is instead a mapping from the set of all possible results of the measurement to $S \times \{-1, 0, 1\}$. For instance, for the result j of measurement, and $\delta(s, \sigma)(j) = (s_j, d)$, then
   (i) if $s_j \in S\backslash(S_{acc} \cup S_{rej})$, with probability $\|P_j|\psi\rangle\|^2$ the updated quantum state is $P_j|\psi\rangle/\|P_j|\psi\|$ and the classical state is s_j together with the tape head moving by means of d;

(ii) if $s_j \in S_{acc}$, with probability $\|P_j|\psi\rangle\|^2$ the machine accepts the input and the computation halts;

(iii) and similarly, if $s_j \in S_{rej}$, with probability $\|P_j|\psi\rangle\|^2$ the machine rejects the input and the computation halts.

It is seen that if the current all possible classical states are in $S_{acc} \cup S_{rej}$, then the computation for the current input string ends.

On the basis of the above definition, we can naturally define the computing process and the probabilities of accepting and rejecting. For any input string $x \in \Sigma^*$, the machine begins with the initial quantum state $|q_0\rangle$ and classical state s_0 and reads the left end-marker ¢. While in terms of $\Theta(s_0, ¢)$, the quantum state is evolved, by means of $\delta(s_0, ¢)$ the classical state is changed and the tape head is moved correspondingly (in accordance with [6], the tape head is not allowed to move left (right) when it points at ¢ (\$)). In each evolution, the corresponding accepting and rejecting probabilities are computed in terms of whether the transformation function δ enters accepting or rejecting states. The computation will end if all classical states entered are in $S_{acc} \cup S_{rej}$. Therefore, similar to the definition of accepting and rejecting probabilities for MM-1qfa's and 2qfa's [23], the accepting and rejecting probabilities $P_{acc}^{(M)}(x)$ and $P_{rej}^{(M)}(x)$ in M for input x are respectively the sums of all accepting probabilities and all rejecting probabilities before the end of the machine for computing input x.

A language L over alphabet Σ^* is called to be recognized by 2qcfa M with bounded error probability ϵ if $\epsilon \in [0, 1/2)$, and

- for any $x \in L$, $P_{acc}^{(M)}(x) \geq 1 - \epsilon$,
- for any $x \in L^c = \Sigma^* \backslash L$, $P_{rej}^{(M)}(x) \geq 1 - \epsilon$.

We say that 2qcfa M recognizes language L over alphabet Σ with one-sided error $\epsilon > 0$ if $P_{acc}^{(M)}(x) = 1$ for $x \in L$, and $P_{rej}^{(M)}(x) \geq 1 - \epsilon$ for $x \in L^c = \Sigma^* \backslash L$.

As were shown by Ambainis and Watrous [6], for any $\epsilon > 0$, 2qcfa's can recognize palindromes $L_{pal} = \{x \in \{a, b\}^* | x = x^R,\}$ and $L_{eq} = \{a^n b^n | n \in \mathbf{N}\}$ with one-sided error probability ϵ, where ϵ can be arbitrarily small. Here we simply describe their computing process for recognizing L_{eq}, and the details are referred to [6]. In their machine (we denote it by M_{eq}), there are only two quantum states, i.e., $Q = \{q_0, q_1\}$. For any input string $x \in \{a, b\}^*$, M_{eq} firstly checks whether or not x is of the form $a^n b^m$ for $n, m \geq 1$. If not, the machines rejects it immediately; otherwise, the machine reads the input symbols from left to right successively. After reading symbol a (or b), the quantum state part that is described by Hilbert space $l_2(Q)$ is performed by rotating unitary transformation U_α (or U_β), where $\alpha = \sqrt{2}\pi$ (and $\beta = -\sqrt{2}\pi$) is the angle rotated. When the tape head reads the right end-marker \$, the machine performs orthogonal measurement:

- If $\#_x(a) \neq \#_x(b)$, where $\#_x(a)$ (and $\#_x(b)$) represents the number of a (and b) in string x, say n a's and m b's, then there is non-zero probability (at least $\frac{1}{2(n-m)^2}$) for measuring $|q_1\rangle$. Therefore, the machine rejects that part of q_1, and with q_0 its tape head is moved to the first input symbol in the left, and then by performing random walk the tape head reaches the right end-marker \$, repeating this action twice and then flipping k (related to ϵ) coins. If all results are not "heads", the machine accepts with probability $\frac{1}{2^k(n+m+1)^2}$; otherwise, with the rest probability the machine recurs to the beginning configuration and then executes a round again. With at most $O((n+m)^4)$ steps, the rejecting probability is bigger than $1-\epsilon$.

- If $\#_x(a) = \#_x(b) = n$, then with certainty the machine's tape head is moved to the first input symbol in the left, and then by performing random walk the tape head reaches the right end-marker \$, repeating this action twice and then flipping k (related to ϵ) coins. If all results are not "heads", the machine accepts with probability $\frac{1}{2^k(n+m+1)^2}$; otherwise, with the rest probability the machine recurs to the beginning configuration and then executes a round again. With at most $O(n^2)$ steps, the accepting probability is bigger than $1-(1-\frac{1}{2^k(n+m+1)^2})^{cn^2}$, that is close to 1 for appropriate constant c.

Basing on this 2qcfa M_{eq} presented above, we may further observe that some another non-regular languages can also be recognized by 2qcfa's with bounded error probabilities in polynomial time, and, we would state them in the following Remarks to conclude this section.

Remark 1. In terms of the 2qcfa M_{eq} above by Ambainis and Watrous [6], the language $\{a^nb_1^na^mb_2^m|n,m\in\mathbf{N}\}$ can also be recognized by some 2qcfa denoted by $M_{eq}^{(2)}$ with one-sided error probability in polynomial time. Indeed, let $M_{eq}^{(2)}$ firstly checks whether or not the input string, say x, is the form $a^{n_1}b_1^{n_2}a^{m_1}b_2^{m_2}$. If not, then x is rejected certainly; otherwise, $M_{eq}^{(2)}$ simulates M_{eq} for deciding whether or not $a^{n_1}b_1^{n_2}$ is in L_{eq}, by using the a in the right of b_1 as the right end-marker \$. If not, then x is rejected; otherwise, this machine continues to simulate M_{eq} for recognizing $a^{m_1}b_2^{m_2}$, in which b_1 is viewed as the left end-marker ¢. If it is accepted, then x is also accepted; otherwise, x is rejected.

Remark 2. For $k\in\mathbf{N}$, let $L_{eq}(k,a)=\{a^{kn}b^n|n\in\mathbf{N}\}$. Obviously, $L_{eq}(1,a)=L_{eq}$. Then, by means of the 2qcfa M_{eq}, $L_{eq}(k,a)$ can be recognized by some 2qcfa, denoted by $M_{eq}(k,a)$, with one-sided error probability in polynomial time. Indeed, $M_{eq}(k,a)$ is derived from M_{eq} by replacing U_β with U_{β_k}, where $\beta_k=\sqrt{2}k\pi$. Likewise, denote $L_{eq}(k,b)=\{b^{kn}a^n|n\in\mathbf{N}\}$. Then $L_{eq}(k,b)$ can be recognized by some 2qcfa $M_{eq}(k,b)$ with one-sided error probability in polynomial time.

Remark 3. Let $L_= = \{x\in\{a,b\}^*|\#_x(a)=\#_x(b)\}$, where $\#_x(a)$ (and $\#_x(b)$) represents the number of a (and b) in string x. Then $L_=$ is recognized by some 2qcfa, denoted by $M_=$, with one-sided error probability in polynomial time. Indeed, by observing the words in $L_=$, $M_=$ can be directly derived from M_{eq} above by omitting the beginning process for checking whether or not the input string is of the form $a^n b^m$.

3. Operation properties of 2qcfa's

This section deals with operation properties of 2qcfa's, and, a number of examples as application are incorporated. For convenience, we use notations $2QCFA_\epsilon(poly-time)$ and $2QCFA(poly-time)$ to denote the classes of all languages recognized by 2qcfa's with given error probability $\epsilon \geq 0$ and with any error probabilities in $[0,1)$, respectively, which run in polynomial expected time; for any language $L \in 2QCFA(poly-time)$, let QS_L and CS_L denote respectively the minimum numbers of quantum states and classical states of the 2qcfa that recognizes L with error probability in $[0,1)$. Firstly, we consider intersection operation.

Theorem 1. If $L_1 \in 2QCFA_{\epsilon_1}(poly-time)$, $L_2 \in 2QCFA_{\epsilon_2}(poly-time)$, then $L_1 \cap L_2 \in 2QCFA_\epsilon(poly-time)$ with $\epsilon = \epsilon_1 + \epsilon_2 - \epsilon_1\epsilon_2$.

Proof. Let M_1 and M_2 be 2qcfa's for recognizing L_1 and L_2 with error probabilities $\epsilon_1, \epsilon_2 \geq 0$, respectively. The basic idea is as follows. Firstly let the machine M constructed simulate M_1. If M_1 rejects, then M also rejects; if M_1 accepts, then M continues to simulate M_2 and then M_2 decides the accepting and rejecting probabilities. This 2qcfa M may be more clearly described by the following process.

For input string x, M_1 and M_2 with initial quantum state $|q_{1,0}\rangle$ and $|q_{2,0}\rangle$ as well as classical state $s_{1,0}$ and $s_{2,0}$, respectively; also, M has initial quantum state $|q_{1,0}\rangle$ and classical state $s_{1,0}$. M firstly simulate M_1. If M_1 rejects, then M rejects; if M_1 accepts, then M becomes quantum state $|q_{2,0}\rangle$ and classical state $s_{2,0}$, and continues to simulate M_2. If M_2 accepts, then also M accepts; otherwise M rejects as M_2 does.

Basing on the analysis above, we now prove this theorem more formally. Let 2qcfa's

$M_i = (Q_i, S_i, \Sigma_i, \Theta_i, \delta_i, q_{i,0}, s_{i,0}, S_{i,acc}, S_{i,rej})$

for accepting L_i with error probabilities $\epsilon_i \geq 0$ $(i = 1, 2)$, where we suppose that for $i = 1, 2$,

- $Q_i = \{q_{i,0}, q_{i,1}, \ldots, q_{i,n_i}\}$,
- $S_i = \{s_{i,0}, s_{i,1}, \ldots, s_{i,m_i}\}$.

We construct 2qcfa $M = (Q, S, \Sigma, \Theta, \delta, q_0, s_0, S_{acc}, S_{rej})$ where:

- $\Sigma = \Sigma_1 \cap \Sigma_2$,
- $q_0 = q_{1,0}$,
- $s_0 = s_{1,0}$,
- $Q = Q_1 \cup Q_2$ (also, we can equivalently use $Q = Q_1 \oplus Q_2$ without essential difference),

- $S = S_1 \cup S_2 \cup \{t^{(1,j)} | j = 0, 1, \ldots, n_1\}$,
- $S_{rej} = S_{1,rej} \cup S_{2,rej}$,
- $S_{acc} = S_{acc,2}$,

and Θ and δ are defined as follows:

1. For any $s \in S_1 \backslash S_{1,acc} \cup S_{1,rej}$, $\sigma \in \Sigma \cup \{¢, \$\}$,

   (i) if $\Theta_1(s, \sigma) \in \mathcal{U}(l_2(Q_1))$, i.e., a unitary operator on $l_2(Q_1)$, then $\Theta(s, \sigma)$ is unitary operator on $l_2(Q)$ by extending $\Theta_1(Q)$ in terms of $\Theta(s, \sigma)|q_{2,j}\rangle = |q_{2,j}\rangle$ for $0 \le j \le n_2$, and $\delta(s, \sigma) = \delta_1(s, \sigma)$;

   (ii) if $\Theta_1(s, \sigma) \in \mathcal{M}(l_2(Q_1))$, i.e., an orthogonal measurement on $l_2(Q_1)$, say the measurement is specified by the set of $\{P_j\}$ of projectors, where each P_j is a projection operator and $\delta_1(s, \sigma) = (s_j, d_j)$, then $\delta(s, \sigma)(j) = \delta_1(s, \sigma)(j)$, and $\Theta(s, \sigma)$ is an orthogonal measurement described by the set $\{P_j^{'}\} \cup \{I_2\}$ of projectors on $l_2(Q) = l_2(Q_1 \cup Q_2)$, where $P_j^{'}$ are projection operators by extending P_j with $P_j^{'}|q_{2,j}\rangle = 0$ for $0 \le j \le n_2$, and I_2 is projection operator mapping to $l_1(Q_2)$, that is, an identity operator on $l_2(Q_2)$ and $I_2|q_{1,j}\rangle = 0$ for $0 \le j \le n_1$.

2. For any $s \in S_{1,acc}$, $\sigma \in \Sigma \cup \{¢, \$\}$,

   (i) if $\sigma \ne ¢$, then $\Theta(s, \sigma) = I$, where I is identity operator on $l_2(Q)$, and $\delta(s, \sigma) = (s, -1)$;

   (ii) if $\sigma = ¢$, then $\Theta(s, \sigma)$ is an orthogonal measurement described by projectors $\{|q_{1,j}\rangle\langle q_{1,j}| | q_{1,j} \in Q_1\}$, $\delta(s, \sigma)(1, j) = (t^{(1,j)}, 0)$; $\Theta(t^{(1,j)}, ¢) = U(q_{1,j}, q_{2,0})$, $\delta(t^{(1,j)}, ¢) = (\mathrm{s}_{2,0}, 0)$, where $U(q_{1,j}, q_{2,0})$ is a unitary operator on $l_2(Q)$ satisfying $U|q_{1,j}\rangle = |q_{2,0}\rangle$.

3. For any $s \in S_2$, $\sigma \in \Sigma \cup \{¢, \$\}$,

   (i) if $\Theta_2(s, \sigma)$ is a unitary operator on $l_2(Q_2)$, then $\Theta(s, \sigma)$ is a unitary operator on $l_2(Q)$ by extending $\Theta_2(s, \sigma)$ with $\Theta(s, \sigma)|q_{1,j}\rangle = |q_{1,j}\rangle$ for $0 \le j \le n_1$, and $\delta(s, \sigma) = \delta_2(s, \sigma)$;

   (ii) if $\Theta_2(s, \sigma)$ is an orthogonal measurement on $l_2(Q_2)$ described by projection operators $\{P_j\}$, then $\Theta(s, \sigma)$ is an orthogonal measurement on $l_2(Q)$ specified by projection operators $\{P_j^{'}\} \cup \{I_1\}$, and $\delta(s, \sigma) = (s_j, d_j)$ if $\delta_2(s, \sigma) = (s_j, d_j)$, where $P_j^{'}$ extend P_j to $l_2(Q)$ by defining $P_j^{'}|q_{1,i}\rangle = 0$ for $0 \le i \le n_1$.

In terms of the 2qcfa M constructed above, for any $x \in \Sigma^*$, we have:

- If $x \in L_1 \cap L_2$, then M accepts x with probability at least

  $(1 - \epsilon_1)(1 - \epsilon_2) = 1 - (\epsilon_1 + \epsilon_2 - \epsilon_1 \epsilon_2)$.

- If $x \notin L_1$, then M rejects x with probability at least $1-\epsilon_1$.
- If $x \in L_1$ but $x \notin L_2$, then M rejects x with probability at least $(1-\epsilon_1)(1-\epsilon_2)$.

□

By means of the proof of Theorem 1, we have the following corollaries 1 and 2.

Corollary 1. If languages L_1 and L_2 are recognized by 2qcfa's M_1 and M_2 with one-sided error probabilities $\epsilon_1, \epsilon_2 \in [0, \frac{1}{2})$ in polynomial time, respectively, then $L_1 \cap L_2$ is recognized by some 2qcfa M with one-sided error probability $\epsilon = \max\{\epsilon_1, \epsilon_2\}$ in polynomial time, that is, for any input string x,

- if $x \in L_1 \cap L_2$, then M accepts x with certainty;
- if $x \notin L_1$, then M rejects x with probability at least $1-\epsilon_1$;
- if $x \in L_1$ but $x \notin L_2$, then M rejects x with probability at least $1-\epsilon_2$.

Example 1. We recall that non-regular language $L_= = \{x \in \{a,b\}^* | \#_x(a) = \#_x(b)\}$. For non-regular language $L_=(pal) = \{y = xx^R | x \in L_=\}$, we can clearly check that $L_=(pal) = L_= \cap L_{pal}$. Therefore, by applying Corollary 1, we obtain that $L_=(pal)$ is recognized by some 2qcfa with one-sided error probability ϵ, since both $L_=$ and L_{pal} are recognized by 2qcfa's with one-sided error probability ϵ [6], where ϵ can be given arbitrarily small.

Corollary 2. If $L_1 \in 2QCFA(poly-time)$, $L_2 \in 2QCFA(poly-time)$, then

1. $QS_{L_1 \cap L_2} \leq QS_{L_1} + QS_{L_2}$;
2. $CS_{L_1 \cap L_2} \leq CS_{L_1} + CS_{L_2} + QS_{L_1}$.

Similar to Theorem 1, we can obtain the union operation of 2qcfa's.

Theorem 2. If $L_1 \in 2QCFA_{\epsilon_1}(poly-time)$ and $L_2 \in 2QCFA_{\epsilon_2}(poly-time)$ for $\epsilon_1, \epsilon_2 \geq 0$, then $L_1 \cup L_2 \in 2QCFA_{\epsilon}(poly-time)$ with $\epsilon = \epsilon_1 + \epsilon_2 - \epsilon_1\epsilon_2$.

Proof. The idea is similar to the proof of Theorem 1. Let L_i be accepted by 2qcfa's M_i with error probabilities ϵ_i $(i = 1,2)$. Then we construct a 2qcfa M as the way in Theorem 1, that is to say, we use M firstly to simulate M_1. If M_1 accepts, then M also accepts; otherwise, M continues to simulate M_2, and the accepting or rejecting of M depends on M_2. The process is more clearly described as follows.

For input string x, M_1 and M_2 with initial quantum state $|q_{1,0}\rangle$ and $|q_{2,0}\rangle$ as well as classical state $s_{1,0}$ and $s_{2,0}$, respectively; also, M has initial quantum state $|q_{1,0}\rangle$ and classical state $s_{1,0}$. M firstly simulate M_1. If M_1 accepts, then M accepts; if M_1 rejects, then M

becomes quantum state $|q_{2,0}\rangle$ and classical state $s_{2,0}$, and continues to simulate M_2. If M_2 rejects, then also M rejects; otherwise M accepts as M_2 does.

Similarly to Theorem 1, for any $x \in \Sigma^*$, we have:

- If $x \in L_1$, then M accepts x with probability at least $1 - \epsilon_1$.
- If $x \notin L_1$, but $x \in L_2$, then M accepts x with probability at least $(1 - \epsilon_1)(1 - \epsilon_2)$.
- If $x \notin L_1$ and $x \notin L_2$, then M rejects x with probability at least $(1 - \epsilon_1)(1 - \epsilon_2)$.

Since the specific process is analogous to Theorem 1, we leave the details out here.

□

Due to the proof of Theorem 2, we also have the following corollary.

Corollary 3. If languages L_1 and L_2 are recognized by 2qcfa's M_1 and M_2 with one-sided error probabilities $\epsilon_1, \epsilon_2 \in [0, \frac{1}{2})$ in polynomial time, respectively, then there exists 2qcfa M such that $L_1 \cup L_2$ is recognized by 2qcfa M with error probability at most $\epsilon_1 + \epsilon_2 - \epsilon_1\epsilon_2$ in polynomial time, that is, for any input string x,

- if $x \in L_1$, then M accepts x with certainty;
- if $x \notin L_1$, but $x \in L_2$, then M accepts x with probability at least $1 - \epsilon_1$;
- if $x \notin L_1$ and $x \notin L_2$, then M rejects x with probability at least $(1 - \epsilon_1)(1 - \epsilon_2)$.

Similar to Corollary 2, we have:

Corollary 4. If $L_1 \in 2QCFA(poly - time)$, $L_2 \in 2QCFA(poly - time)$, then

- $QS_{L_1 \cup L_2} \leq QS_{L_1} + QS_{L_2}$;
- $CS_{L_1 \cup L_2} \leq CS_{L_1} + CS_{L_2} + QS_{L_1}$.

Example 2. As indicated in Remark 2, $L_{eq}(k, a) = \{a^{kn}b^n | n \in \mathbf{N}\}$ and $L_{eq}(k, b) = \{b^n a^n | n \in \mathbf{N}\}$ are recognized by 2qcfa's with one-sided error probabilities (as demonstrated by Ambainis and Watrous [6], these error probabilities can be given arbitrarily small) in polynomial time. Therefore, by using Corollary 3, we have that for any $m \in \mathbf{N}$, $\cup_{k=1}^{m} L_{eq}(k, a)$ and $\cup_{k=1}^{m} L_{eq}(k, b)$ are recognized by 2qcfa's with error probabilities in $[0, \frac{1}{2})$ in polynomial time.

For language L over alphabet Σ, the complement of L is $L^c = \Sigma^* \backslash L$. For the class of languages recognized by 2qcfa's with bounded error probabilities, the unary complement operation is also closed.

Theorem 3. If $L \in 2QCFA_\epsilon(poly{-}time)$ for error probability ϵ, then $L^c \in 2QCFA_\epsilon(poly{-}time)$.

Proof. Let 2qcfa $M = (Q, S, \Sigma, \Theta, \delta, q_0, s_0, S_{acc}, S_{rej})$ accept L with error probability $\epsilon \in [0, \frac{1}{2})$. Then we can construct 2qcfa M^c only by exchanging the classical accepting and rejecting states in M, that is, $M^c = (Q, S, \Sigma, \Theta, \delta, q_0, s_0, S^c_{acc}, S^c_{rej})$ where $Q, S, \Sigma, \Theta, \delta, q_0, s_0$ are the same as those in M, and, $S^c_{acc} = S_{rej}$, $S^c_{rej} = S_{acc}$. Clearly, L^c is accepted by M^c with error probability ϵ.

$\square$

From the proof of Theorem 3 it follows Corollary 5.

Corollary 5. If $L \in 2QCFA(poly - time)$, then

- $QS_{L^c} = QS_L$;
- $CS_{L^c} = CS_L$.

Example 3. For non-regular language $L_=$, its complement $L^c_= = \{x \in \{a, b\}^* | \#_x(a) \neq \#_x(b)\}$ is recognized by 2qcfa with bounded error probability in polynomial expected time, by virtue of Remark 3 and Theorem 3.

For language L over alphabet Σ, the reversal of L is $L^R = \{x^R | x \in L\}$ where x^R is the reversal of x, i.e., if $x = \sigma_1\sigma_2 \ldots \sigma_n$ then $x^R = \sigma_n\sigma_{n-1} \ldots \sigma_1$. For $2QCFA_\epsilon(poly - time)$ with $\epsilon \in [0, 1/2)$, the reversal operation is closed.

Theorem 4. If $L \in 2QCFA_\epsilon(poly - time)$, then $L^R \in 2QCFA_\epsilon(poly - time)$.

Proof. Let L be recognized by a 2qcfa M with error probability $\epsilon \in [0, \frac{1}{2})$. Then we can construct a 2qcfa M^c simulate M from the converse direction of the tape head moving. More specifically, suppose $M = (Q, S, \Sigma, \Theta, \delta, q_0, s_0, S_{acc}, S_{rej})$. Then, we construct $M^R = (Q^R, S^R, \Sigma, \Theta^R, \delta^R, q^R_0, s^R_0, S^R_{acc}, S^R_{rej})$ where $Q^R = Q \cup \{q^R_0\}$, $S^R = S \cup \{s^R_0\}$ with $q^R_0 \notin Q$, $s^R_0 \notin S$, $S^R_{acc} = S_{acc}$, $S^R_{rej} = S_{rej}$, Θ^R and δ^R are defined as follows.

1. For $\sigma \in \Sigma \cup \{¢\}$, $\Theta^R(s^R_0, \sigma) = I$, where I is identity operator on $l_2(Q^R)$, $\delta^R(s^R_0, \sigma) = (s^R_0, 1)$; and $\Theta^R(s^R_0, \$) = U(q^R_0, q_0)$, where $U(q^R_0, q_0)$ is a unitary operator on $l_2(Q^R)$ satisfying $U(q^R_0, q_0)|q^R_0\rangle = |q_0\rangle$, and $\delta^R(s^R_0, \$) = (s_0, 0)$.

2. For $s \in S$, $\sigma \in \Sigma \cup \{¢, \$\}$, if $\Theta(s, \sigma)$ is a unitary operator on $l_2(Q)$, then $\Theta^R(s, \sigma)$ is also unitary operator on $l_2(Q^R)$ by extending $\Theta(s, \sigma)$ with $\Theta^R(s, \sigma)|q^R_0\rangle = |q^R_0\rangle$ and $\Theta^R(s, \sigma)|\phi\rangle = \Theta(s, \sigma)|\phi\rangle$ for $|\phi\rangle \in l_2(Q)$, and $\delta^R(s, \sigma) = (s', -d)$ if $\delta(s, \sigma) = (s', d)$.

3. For $s \in S$, $\sigma \in \Sigma \cup \{¢, \$\}$, if $\Theta(s, \sigma)$ is an orthogonal measurement on $l_2(Q)$ described by projectors $\{P_j\}$, then $\Theta^R(s, \sigma)$ is also an orthogonal measurement on $l_2(Q^R)$ described

by projectors $\{P_j'\}\cup\{I_2\}$, where P_j' extend P_j to $l_2(Q^R)$ by defining $P_j'|q_0^R\rangle=0$, and projection operator I_2 mapping to $l_2(\{q_0^R\})$.

Then, in terms of the 2qcfa M^R constructed above, M^R accepts L^R with bounded error probability ϵ.

□

By means of the proof of Theorem 4 we clearly obtain the following corollary.

Corollary 6. If $L\in 2QCFA(poly-time)$, then

- $QS_L-1\leq QS_{L^R}\leq QS_L+1$;
- $CS_L-1\leq CS_{L^R}\leq CS_L+1$.

For languages L_1 and L_2 over alphabets Σ_1 and Σ_2, respectively, the catenation of L_1 and L_2 is $L_1L_2=\{x_1x_2|x_1\in\Sigma_1,x_2\in\Sigma_2\}$. We do not know whether or not the catenation operation in $2QCFA_\epsilon$ is closed, but under certain condition we can prove that the catenation of two languages in $2QCFA_\epsilon$ is closed.

Theorem 5. Let $L_i\in 2QCFA_\epsilon$, and $\Sigma_1\cap\Sigma_2=\emptyset$ where Σ_i are alphabets of L_i $(i=1,2)$. Then the catenation L_1L_2 of L_1 and L_2 is also recognized by a 2qcfa with error probability at most $\epsilon=\epsilon_1+\epsilon_2-\epsilon_1\epsilon_2$.

Proof. Let ε denote empty string. We may consider four cases:

1. $\varepsilon\notin L_1\cup L_2$;
2. $\varepsilon\notin L_2$ but $\varepsilon\in L_1$;
3. $\varepsilon\notin L_1$ but $\varepsilon\in L_2$;
4. $\varepsilon\in L_1\cap L_2$.

Here we only prove case 1, since the other cases are similar.

Suppose that L_i are recognized by 2qcfa's $M_i=(Q_i,S_i,\Sigma_i,\Theta_i,\delta_i,q_{i,0},s_{i,0},S_{i,acc},S_{i,rej})$, with error probabilities ϵ_i $(i=1,2)$. Then we construct 2qcfa M accepting L_1L_2 with error probability $\epsilon=\epsilon_1+\epsilon_2-\epsilon_1\epsilon_2$. Firstly we let M check whether or not the input is the form of $xy\in\Sigma_1^+\Sigma_2^+$ where Σ_i^+ denote the set of all non-empty strings over Σ_i; otherwise M rejects the input immediately. Then let M simulate M_1, and, as soon as M_1 meets an input symbol not in Σ_1, M_1 views this input symbol as \$. Therefore, if M_1 rejects the first part of input string, then M rejects the input string; otherwise, M continues to compute the second part of the input string by simulating M_2, and, therefore, the results of rejecting and accepting of M further depend on M_2. Hence, the computing process of M is roughly as follows.

For input string x, M checks whether x is the form in $\Sigma_1^+\Sigma_2^+$. If it is not such a form, then M rejects it; otherwise M continues to simulate M_2, for resulting in the accepting and rejecting probabilities.

More formally, let $M_i = (Q_i, S_i, \Sigma_i, \Theta_i, \delta_i, q_{i,0}, s_{i,0}, S_{i,acc}, S_{i,rej})$ $(i = 1, 2)$. Then $M = (Q, S, \Sigma, \Theta, \delta, q_0, s_0, S_{acc}, S_{rej})$, where:

- $Q = Q_1 \cup Q_2$,
- $S = S_1 \cup S_2 \cup \{s_0, s_1, s_2, s_3\}$ with $\{s_0, s_1, s_2, s_3\} \cap (S_1 \cup S_2) = \emptyset$,
- $q_0 = q_{1,0}$,
- $S_{acc} = S_{acc,2}$,
- $S_{rej} = S_{1,rej} \cup S_{2,rej} \cup \{s_2\}$,

and Θ and δ are defined as follows.

1. Firstly, let M check the form of the input string,
   (i) for any $\sigma \in \Sigma_1 \cup \{¢\}$, $\Theta(s_0, \sigma) = I$, where I is identity operator on $l_2(Q)$, $\delta(s_0, \sigma) = (s_0, 1)$;
   (ii) for any $\sigma \in \Sigma_1$, $\Theta(s_1, \sigma) = I$, $\delta(s_1, \sigma) = (s_2, 0)$, where $s_2 \in S_{rej}$;
   (iii) for any $\sigma \in \Sigma_2$, $\Theta(s_0, \sigma) = I$, $\delta(s_0, \sigma) = (s_1, 1)$;
   (iv) for $\sigma = \$$, $\Theta(s_1, \$) = I$, $\delta(s_1, \$) = (s_3, -1)$;
   (v) for $\sigma \in \Sigma_1 \cup \Sigma_2$, $\Theta(s_3, \sigma) = I$, $\delta(s_3, \sigma) = (s_3, -1)$;
   (vi) for $\sigma = ¢$, $\Theta(s_3, \sigma) = I$, $\delta(s_3, \sigma) = (s_{1,0}, 0)$.
2. Secondly, let M simulate M_1. For $\sigma \in \Sigma_1 \cup \{¢\}$, $s \in S_1$,
   (i) if $\Theta_1(s, \sigma)$ is a unitary operator on $l_2(Q_1)$, then $\Theta(s, \sigma)$ is also a unitary operator on $l_2(Q)$ by extending $\Theta_1(s, \sigma)$ with $\Theta(s, \sigma)|q\rangle = |q\rangle$ for $q \in Q_2$ and $\Theta(s, \sigma)|\phi\rangle = \Theta_1(s, \sigma)|\phi\rangle$ for $|\phi\rangle \in l_2(Q_1)$, and $\delta(s, \sigma) = \delta_1(s, \sigma)$;
   (ii) if $\Theta_1(s, \sigma)$ is an orthogonal measurement described by projectors $\{P_j\}$, then $\Theta(s, \sigma)$ is also an orthogonal measurement specified by projection operators $\{P_j^{'}\} \cup \{I_2\}$, where $P_j^{'}$ are the extensions of P_j to $l_2(Q)$ by defining $P_j^{'}|q\rangle = 0$ for $q \in Q_2$, and I_2 is identity operator on $l_2(Q_2)$, and $I_2|q\rangle = 0$ for $q \in Q_1$; on the other hand, the definition of $\delta(s, \sigma)$ is in terms of $\delta_1(s, \sigma)$, i.e., $\delta(s, \sigma)$ maps the measuring result of $P_j^{'}$ to the same element as $\delta_1(s, \sigma)(j)$, and $\delta(s, \sigma)$ maps the measuring result of I_2 to any classical state and direction (indeed, before measuring, the quantum superposition state does not include Q_2, and, therefore, the probability of obtaining measuring result by performing operator I_2 is zero).

3. For $\sigma \in \Sigma_2$, $s \in S_1$, $\Theta(s,\sigma) = \Theta_1(s,\$)$ and $\delta(s,\sigma) = \delta_1(s,\$)$.

4. For $s \in S_{1,acc}$,

   (i) if $\sigma \in \Sigma_1 \cup \{¢\}$, then $\Theta(s,\sigma) = I$, where I is identity operator on $l_2(Q)$, and $\delta(s,\sigma) = (s, 1)$;

   (ii) if $\sigma \in \Sigma_2$, then $\Theta(s,\sigma) = I$, where I is identity operator on $l_2(Q)$, and $\delta(s,\sigma) = (s_{2,0}, -1)$.

5. For $\sigma \in \Sigma_1$,

   (i) if $\Theta_2(s_{2,0}, ¢)$ is a unitary operator on $l_2(Q_2)$, then $\Theta(s_{2,0},\sigma)$ is also a unitary operator on $l_2(Q)$ by directly extending $\Theta_2(s_{2,0}, ¢)$, and $\delta(s_{2,0},\sigma) = \delta_2(s_{2,0}, ¢)$;

   (ii) if $\Theta_2(s_{2,0}, ¢)$ is an orthogonal measurement on $l_2(Q_2)$ described by projectors $\{P_j\}$, then $\Theta(s_{2,0},\sigma)$ is also an orthogonal measurement on $l_2(Q)$ specified by $\{P_j^{'}\} \cup \{I_1\}$ where $P_j^{'}$ are the extensions of P_j to $l_2(Q)$ by defining $P_j^{'}|q\rangle = 0$ for $q \in Q_1$, and I_1 is defined as $I_1|q\rangle = \begin{cases} |q\rangle, & q \in Q_1, \\ 0, & q \in Q_2, \end{cases}$ and, as above, $\delta(s_{2,0},\sigma)(j) = \delta(s_{2,0}, ¢)(j)$.

6. For $\sigma \in \Sigma_2 \cup \{\$\}$ and $s \in S_2$, $\Theta(s,\sigma)$ and $\delta(s,\sigma)$ are defined by means of $\Theta_2(s,\sigma)$ and $\delta_2(s,\sigma)$ in the light of Case 5 above.

According to the 2qcfa M specified above, for any $x \in (\Sigma_1 \cup \Sigma_2)^*$, we have:

- If x is not in $\Sigma_1^+\Sigma_2^+$, then x is rejected with certainty.

- If x is in $\Sigma_1^+\Sigma_2^+$, say $x = x_1x_2$ for $x_i \in \Sigma_i^+$ $(i = 1, 2)$, then a) if $x_1 \notin L_1$, then x is rejected with probability at least $1 - \epsilon_1$, b) and if $x_1 \in L_1$ and $x_2 \notin L_2$, then x is rejected with probability at least $(1 - \epsilon_1)(1 - \epsilon_2)$.

- If $x \in L_1L_2$, then x is accepted by M with probability at least $(1 - \epsilon_1)(1 - \epsilon_2)$.

$\square$

From Theorem 5 it follows the following corollary.

Corollary 7. Let languages L_i over alphabets Σ_i be recognized by 2qcfa's with one-sided error probabilities ϵ_i $(i = 1, 2)$ in polynomial time. If $\Sigma_1 \cap \Sigma_2 = \emptyset$, then the catenation L_1L_2 is recognized by some 2qcfa with one-sided error probability $\max\{\epsilon_1, \epsilon_2\}$, in polynomial time.

Remark 4. As indicated in Remark 1, the catenation, $\{a^nb_1^na^mb_2^m | n, m \in \mathbf{N}\}$, of $L_{eq}^{(1)} = \{a^nb_1^n | n \in \mathbf{N}\}$ and $L_{eq}^{(2)} = \{a^nb_2^n | n \in \mathbf{N}\}$, can also be recognized by some 2qcfa with one-sided error probability ϵ in polynomial time, where ϵ can be arbitrarily small. Therefore, in Theorem 5, the condition of $\Sigma_1 \cap \Sigma_2 = \emptyset$ is not necessary.

4. Concluding remarks

2qcfa's were introduced by Ambainis and Watrous [6], and this kind of computing models with classical tape heads is more restricted than the usual 2qfa's [23], but it is still more powerful than 2pfa's. As a continuation of [6], in this note, we have dealt with a number of operation properties of 2qcfa's. We proved that the Boolean operations (intersection, union, and complement) and the reversal operation of the class of languages recognized by 2qcfa's with error probabilities are closed; as corollaries, we showed that the intersection, complement, and reversal operations in the class of languages recognized by 2qcfa's with one-sided error probabilities (in $[0, \frac{1}{2})$) are closed. Furthermore, we verified that the catenation operation in the class of languages recognized by 2qcfa's with error probabilities is closed under certain restricted condition (this result also holds for the case of one-sided error probabilities belonging to $[0, \frac{1}{2})$). As well, the numbers of states of these 2qcfa's for the above operations were presented, and some examples were included for an application of the derived results. For instance, $\{xx^R|x \in \{a,b\}^*, \#_x(a) = \#_x(b)\}$ was shown to be recognized by 2qcfa with one-sided error probability $0 \leq \epsilon < \frac{1}{2}$ in polynomial time.

These operation properties presented may apply to 2qfa's [23], but the unitarity should be satisfied in constructing 2qfa's, and, therefore, more technical methods are likely needed or we have to add some restricted conditions (for example, we may restrict the initial state not to be entered again). On the other hand, in Corollaries 2 and 4, the lower bounds need be further fixed. We would like to further consider them in the future.

Acknowledgement

I would like to thank Dr. Tomoyuki Yamakami for helpful discussion regarding quantum automata.

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