LECTURE 4 Wednesday

1 Kronecker Multiplication

can be applied to any size and shape of matrices.

here are examples

$$\begin{bmatrix} a \\ b \end{bmatrix} \otimes \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a\begin{bmatrix} c \\ d \end{bmatrix} \\ b\begin{bmatrix} c \\ d \end{bmatrix} \end{bmatrix} = \begin{bmatrix} ac \\ ad \\ bc \\ bd \end{bmatrix}$$

$$|0\rangle = 0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$|1\rangle = 1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$|0\rangle \otimes |1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1\begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 0\begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{array}{c} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \end{array} \begin{array}{l} 00 \\ 01 \\ 10 \\ 11 \end{array}$$

$$\begin{bmatrix} a & b \end{bmatrix} \otimes \begin{bmatrix} c & d \end{bmatrix} = \begin{bmatrix} a\begin{bmatrix} c & d \end{bmatrix} & b\begin{bmatrix} c & d \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} ac & ad & bc & bd \end{bmatrix}$$

$$\begin{bmatrix} a & b \end{bmatrix} \otimes \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a\begin{bmatrix} c \\ d \end{bmatrix} & b\begin{bmatrix} c \\ d \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} ac & bc \\ ad & bd \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} \otimes \begin{bmatrix} c & d \end{bmatrix} = \begin{bmatrix} a\begin{bmatrix} c & d \end{bmatrix} \\ b\begin{bmatrix} c & d \end{bmatrix} \end{bmatrix} = \begin{bmatrix} ac & ad \\ bc & bd \end{bmatrix}$$

$*$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} * \begin{bmatrix} x & y \\ z & v \end{bmatrix} = \begin{bmatrix} ax+bz & ay+bw \\ cx+dz & cy+dv \end{bmatrix}$$

② convolution operator on vectors

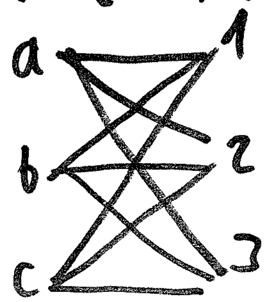
$$\begin{bmatrix} a & b \end{bmatrix} \text{ conv } \begin{bmatrix} c & d \end{bmatrix}$$

$$(ax+b)\cdot(cx+d) = acx^2 + axd + bcx + bd$$

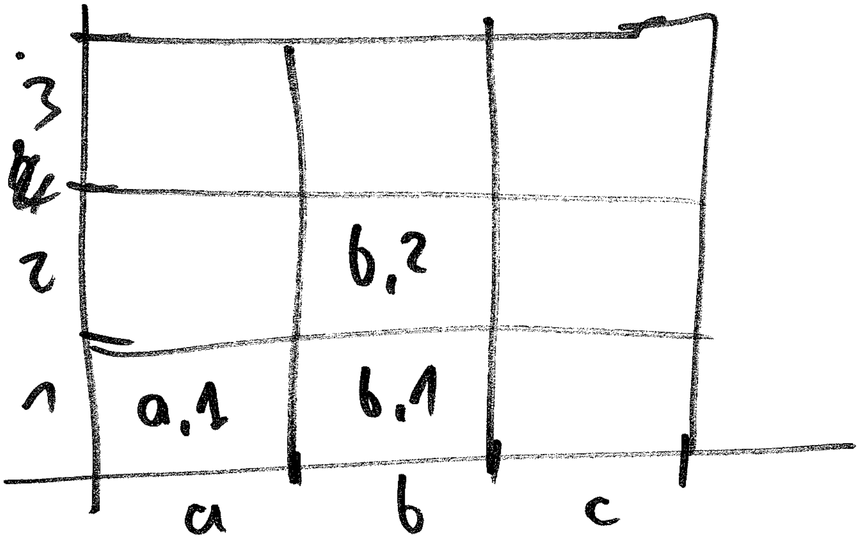
$$= (ac)x^2 + (ad+bc)x + bd$$

$$\begin{bmatrix} ac \,;\, ad+bc \,;\, bd \end{bmatrix}$$

③ CARTESIAN PRODUCT OF SETS ③



Lullus



CARTESIAN PRODUCT.

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \text{ CART } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = [\, ax \;\; ay \;\; az \;\; bx \;\; by \;\; bz \;\; cx \;\; cy \;\; cz \,]$$

4 Reed-Muller forms of Boolean functions

a, b, c — 3 variables

1 a b c

ab ac bc

abc

these are standard trivial (or basis) functions

1	a	b	c	ab	ac	bc	abc

order of coeffic

f_1

1	0	1	1	0	1	0	0

this vector represents PPRM

$1 \oplus b \oplus c \oplus ac$

f_2

1	1	1	0	1	1	0	1

exoring binary vectors creates other functions as vector

$f_1 \oplus f_2$

0	1	0	1	1	0	0	1

another order for PPRM

$$\begin{bmatrix}1\\a\end{bmatrix}\otimes\begin{bmatrix}1\\b\end{bmatrix}\otimes\begin{bmatrix}1\\c\end{bmatrix}$$

$\downarrow$

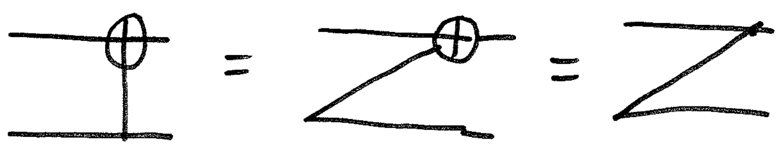
$$\begin{bmatrix}1\begin{bmatrix}1\\b\end{bmatrix}\\ a\begin{bmatrix}1\\b\end{bmatrix}\end{bmatrix}=\begin{bmatrix}1\\b\\a\\ab\end{bmatrix}$$

$$\begin{bmatrix}1\\b\\a\\ab\end{bmatrix}\otimes\begin{bmatrix}1\\c\end{bmatrix}=\begin{bmatrix}1\begin{bmatrix}1\\c\end{bmatrix}\\ b\begin{bmatrix}1\\c\end{bmatrix}\\ a\begin{bmatrix}1\\c\end{bmatrix}\\ ab\begin{bmatrix}1\\c\end{bmatrix}\end{bmatrix}=\begin{bmatrix}1\\c\\b\\bc\\a\\ac\\ab\\abc\end{bmatrix}\begin{bmatrix}0\\0\\0\\0\\0\\1\\0\\1\end{bmatrix}$$

we use Kronecker product of vectors to find order of basis functions

$f = ac \oplus abc$

(5) Butterflies, Kernels, and matrices

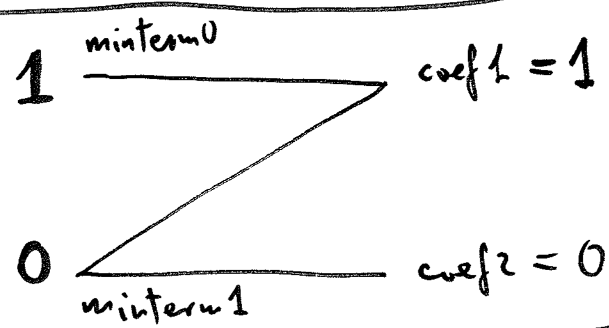


Kernel

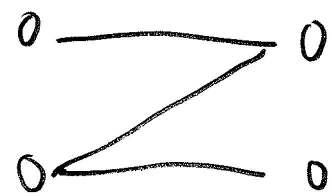
$$\begin{array}{ccc} 0 & 1 & \\ & & \\ \end{array}$$

$$\begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 1 & 1 \\ 1 & 0 & 1 \end{array}$$

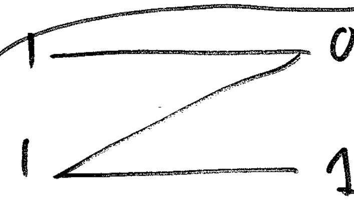
other variant $\begin{bmatrix}0&1\\1&1\end{bmatrix}$!



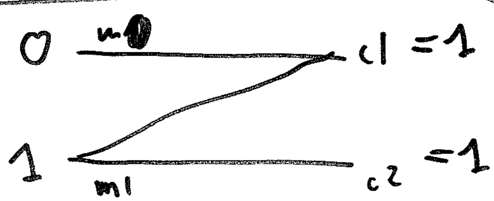
$\begin{bmatrix}1&1\\0&1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$



$\begin{bmatrix}1&1\\0&1\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

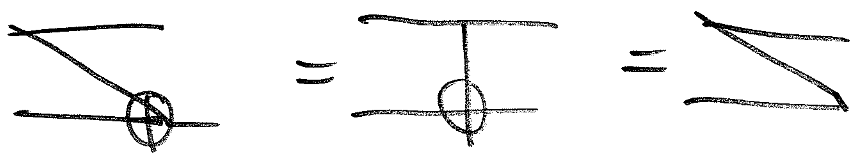


$\begin{bmatrix}1&1\\0&1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}1\oplus1\\0\oplus1\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}$



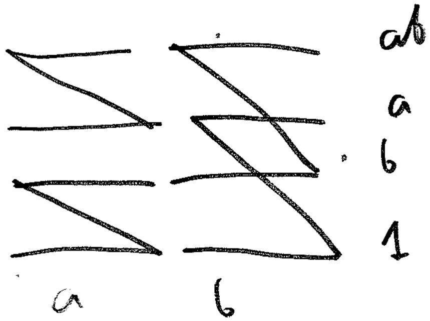
$\begin{bmatrix}1&1\\0&1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}0\oplus1\\0\oplus1\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}$

$$\left.\begin{array}{l} 10 \to 10 \\ 01 \to 11 \\ 00 \to 00 \\ 11 \to 01 \end{array}\right\}\text{ reversible circuit}$$



$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ ← other variant

the variants affect
only order of coefficients,
not their set



← butterfly circuit

for 2 variables will require
$2 \times 2 = 4$ rows and
4 columns
4×4 matrix

$a < \begin{matrix} 0 \\ 1 \end{matrix}$ $b < \begin{matrix} 0 \\ 1 \end{matrix}$

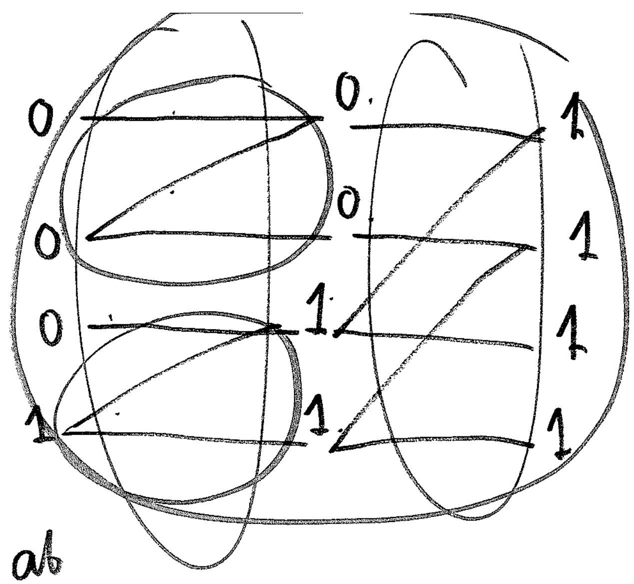
$ab < \begin{matrix} 00 \\ 01 \\ 10 \\ 11 \end{matrix}$

---

remember, this
is a reversible circuit

$2^4 \times 2^4$ matrix

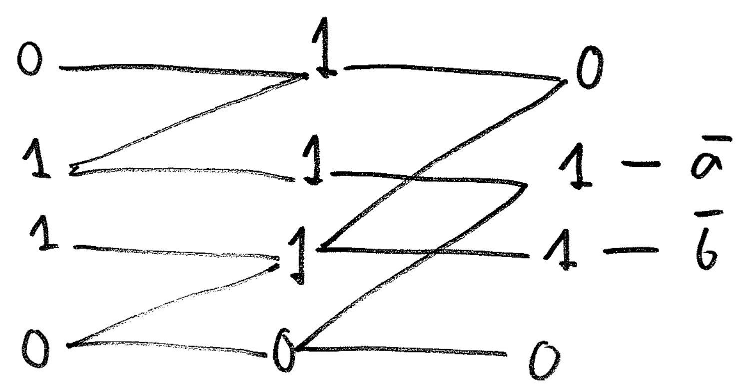
16×16



$\bar{a}\bar{b} \oplus \bar{a} \oplus \bar{b} \oplus 1$ (8)

$= ab$

ab



$\bar{a} \oplus \bar{b} = a \oplus b =$

$\bar{a}b + a\bar{b}$

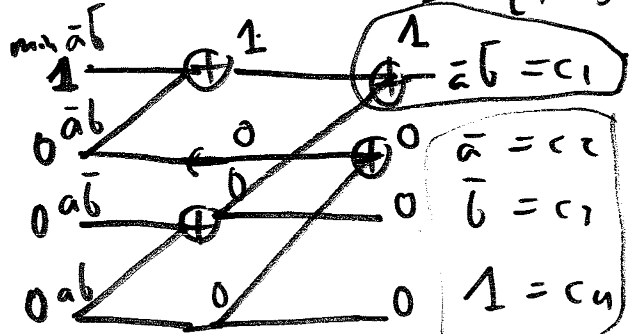
we propagate values
from left to right
through the network
(butterfly)

NPRM transform for two variables, a, b.

$$\begin{array}{cccc} \bar{a}\bar{b} & \bar{a}b & a\bar{b} & ab \end{array}$$
$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} \bar{a}\bar{b} \\ \bar{a} \\ \bar{b} \\ 1 \end{bmatrix}$$

negative polarity.

$$\begin{bmatrix} \bar{a} \\ 1 \end{bmatrix} \times \begin{bmatrix} \bar{b} \\ 1 \end{bmatrix} = \begin{bmatrix} \bar{a}\begin{bmatrix} \bar{b} \\ 1 \end{bmatrix} \\ 1\begin{bmatrix} \bar{b} \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \bar{a}\bar{b} \\ \bar{a} \\ \bar{b} \\ 1 \end{bmatrix}$$



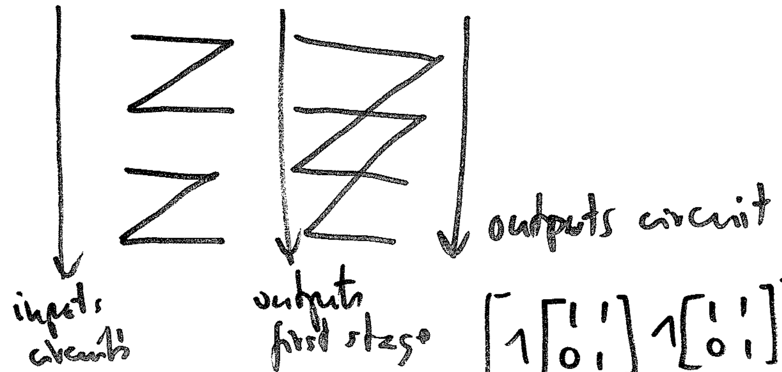
Remember.
Matrices,
Butterflies
and circuits
are equivalent

a \ b	0	1
0	1	0
1	0	0

$\bar{a}\bar{b}$ = BF boolean function

$\bar{a}\bar{b}$ = NPRM its negative polarity RM form

You can use small matrices of transform (with exoring) or large matrices of reversible gates to get the same results, but using small matrices is faster!



$$\begin{bmatrix}1&1\\0&1\end{bmatrix}\otimes\begin{bmatrix}1&1\\0&1\end{bmatrix}=\begin{bmatrix}1\begin{bmatrix}1&1\\0&1\end{bmatrix} & 1\begin{bmatrix}1&1\\0&1\end{bmatrix}\\ 0\begin{bmatrix}1&1\\0&1\end{bmatrix} & 1\begin{bmatrix}1&1\\0&1\end{bmatrix}\end{bmatrix}=\begin{array}{cccc}\bar a\bar b & \bar a b & a\bar b & ab\\ 00 & 01 & 10 & 11\\ \multicolumn{4}{c}{\begin{bmatrix}1&1&1&1\\0&1&0&1\\0&0&1&1\\0&0&0&1\end{bmatrix}}\end{array}$$

coefficients

← calculation of matrix of whole RM transform using Kronecker product

$$\begin{bmatrix}1&1\\0&1\end{bmatrix}\overset{\bar a}{\begin{bmatrix}1\\0\end{bmatrix}}=\begin{bmatrix}1\\0\end{bmatrix}=\bar a$$

$$\begin{bmatrix}1&1\\0&1\end{bmatrix}\overset{a}{\begin{bmatrix}0\\1\end{bmatrix}}=\begin{bmatrix}1\\1\end{bmatrix}\begin{matrix}\leftarrow \bar a\\ \leftarrow 1\end{matrix}$$

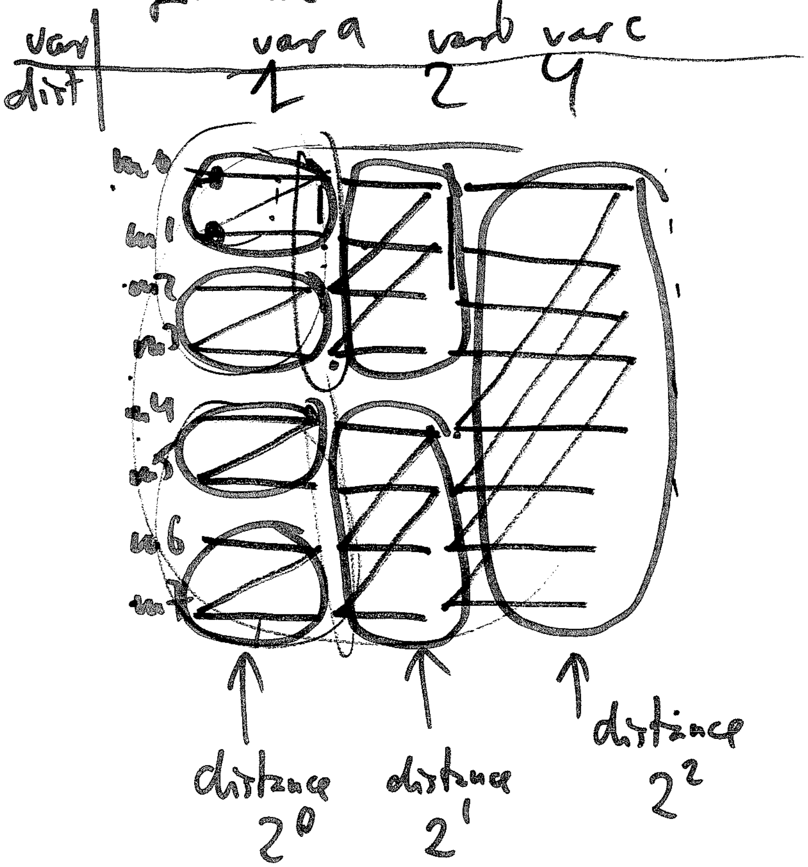
check
$\bar a \oplus 1 = a$

$$\begin{bmatrix}1&1\\0&1\end{bmatrix}\overset{1}{\begin{bmatrix}1\\1\end{bmatrix}}=\begin{bmatrix}1\cdot 1\oplus 1\cdot 1\\ 0\cdot 1\oplus 1\cdot 1\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}\begin{matrix}\leftarrow \bar a\\ \leftarrow 1\end{matrix}$$

$1,\ \bar a$ NPRM

You (as me) may have trouble with order of coefficients. But you can always check!!

How to expand our butterfly for more variables



We showed a regular method of expand butterflies to any number of variables.

Similar methods are used for Walsh, Hadamard, Haar, Fourier and other transforms.

This is much used in DSP and image processing circuits and systems

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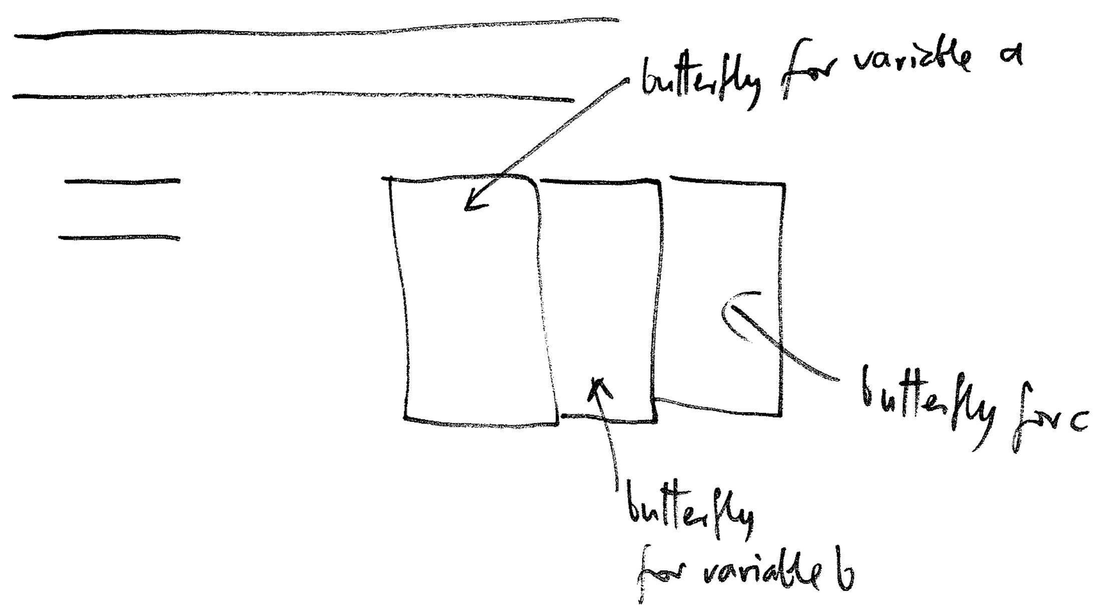
<u>Observation</u>

Every possible FPRM form can be calculated by a combination of stages of butterfly diagram for $\bar{a}$ a



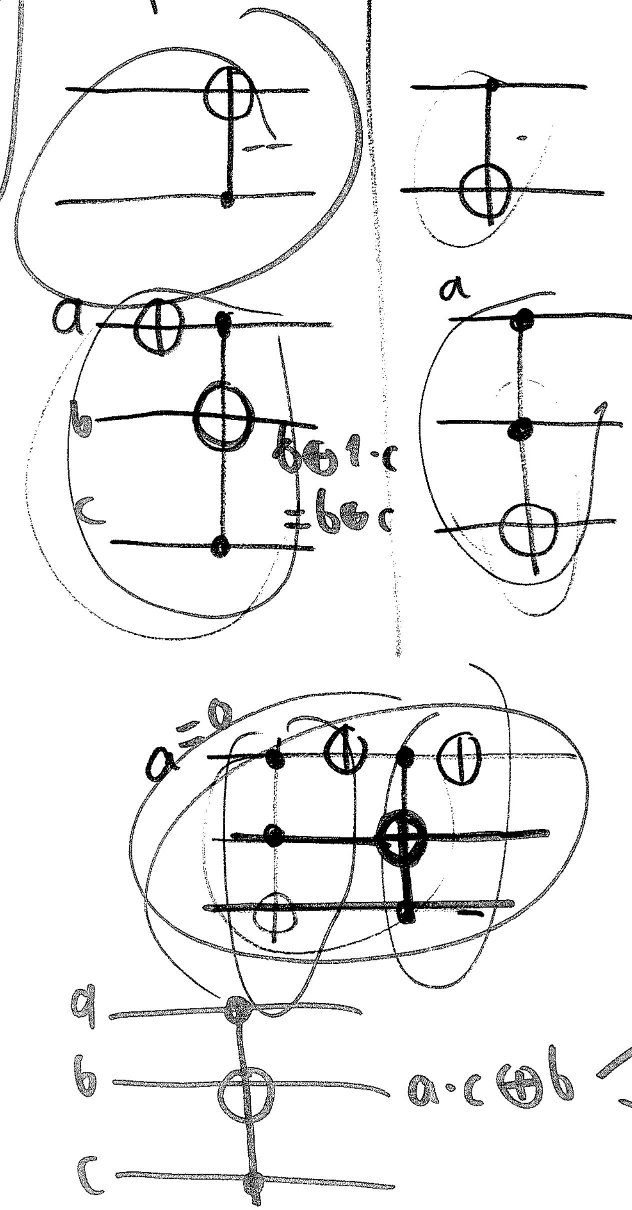
<u>Idea</u> Switch butterfly columns from positive to negative polarity in all possible ways $\longrightarrow 2^n$

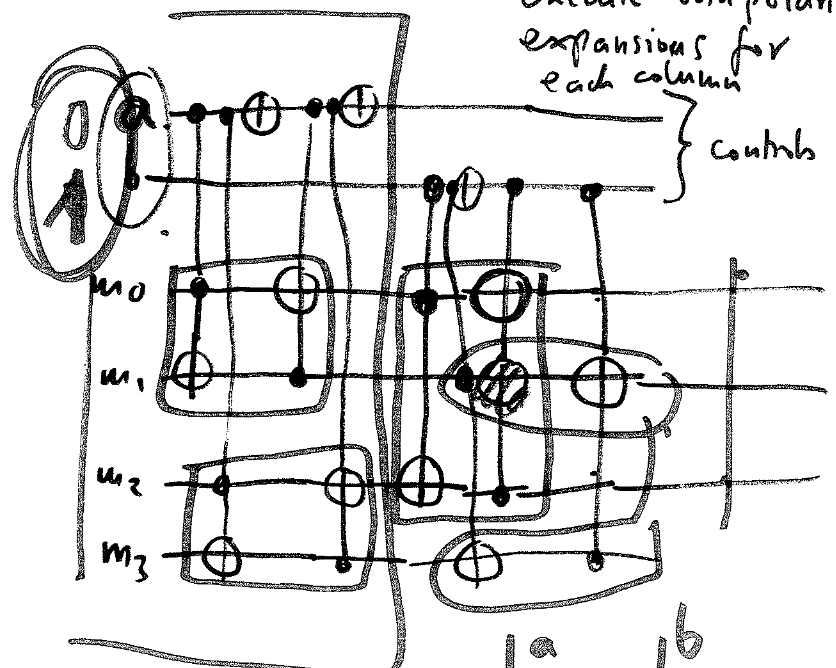
Universal FPRM circuit (6)

for $a=0$ | for $a=1$

$b \oplus 0 \cdot c = b$

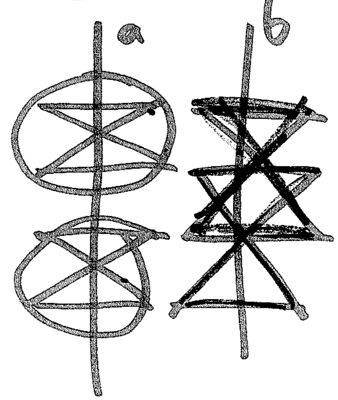
synthesis of circuit that is controlled to execute both polarity expansions for each column (13)





$a=0 \rightarrow b$

$a=1 = c \oplus b$



Please analyse this circuit carefully!

a
b ← controls
c

000
001
010

↗ 3 wires
2^3 states ⟶

2^3 minterms $= 8$

m_0
m_1
m_2
m_3
m_4
m_5
m_6
m_7

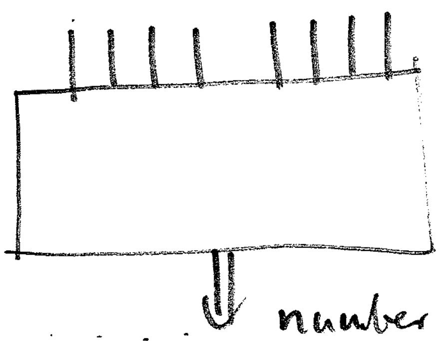
7) Last part of homework 3

Design a complete circuit for all FPRMs of 3 variables that finds the FPRM (one out of $2^3 = 8$) polarity with the minimum number of coefficients = 1

→ you will have to calculate for each polarity how many coefficients on output are equal "1"!

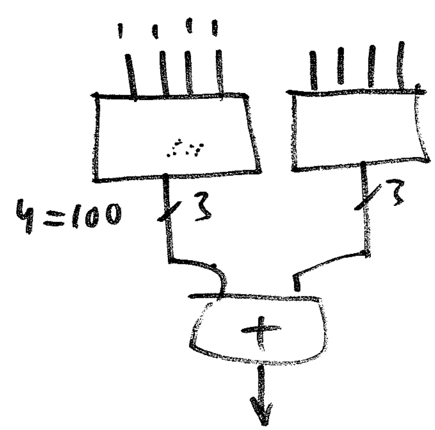
(8) Part of your homework / You will have to design the circuit that is called "number of ones".

(15)



number of inputs that are equal "1"

1000



(16)

Circuit for ~~count of~~ "number of ones" in 4 bits

ab \ cd	00	01	11	10
00	0	1	2	1
01	1	2	3	2
11	2	3	4	3
10	1	2	3	2

# of ones for every cell

ab \ cd	00	01	11	10
00	000	001	010	001
01	001	010	011	010
11	010	011	100	011
10	001	010	011	010

This is a good example x,y,z of a symmetric function

$\underline{bd} + \underline{cd} + \underline{bc}$

$+ \underline{ac} + ad + \underline{ab}$

$b(d+c+a)$

$d(c+a) + ab$

I have to "subtract" this zero

x

0	0	0	0
0	0	0	0
0	0	1	0
0	0	0	0

y

bd, cd, bc, ac, ab, ad

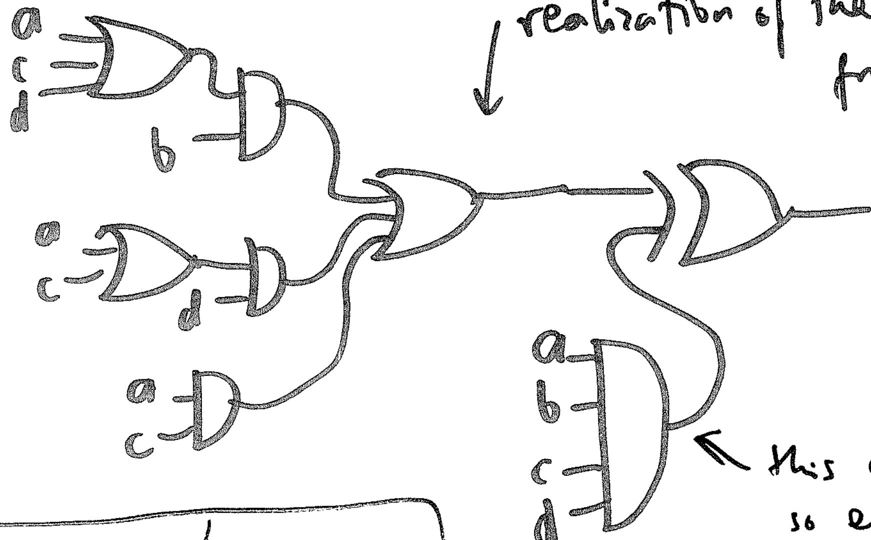
0	0	1	0
0	1	1	1
1	1		1
0	1	1	1

$x = abcd$

z

0	1	0	1
1	0	1	0
0	1	0	1
1	0	1	0

$z = a \oplus b \oplus c \oplus d$



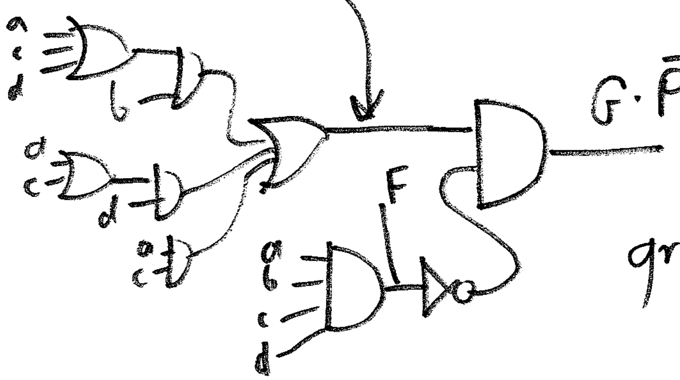
realization of the circuit from page 16 using exoring

this group is included so exoring is the same as sharping

sharp

$$G \# F = G \cdot \overline{F}$$

$\bigcirc$ General inhibition method.



graphical subtraction

(9) The method of inhibiting or subtracting

Graphical synthesis, in software uses sharp

I subtract (create zeros) for only this part of G that is included in F

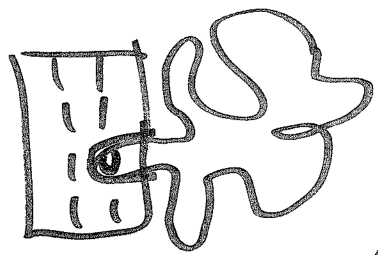
ab \ cd	00	01	11	10
00	0	1	1	0
01	0	1	1	0
11	0	1	0	0
10	0	1	1	0

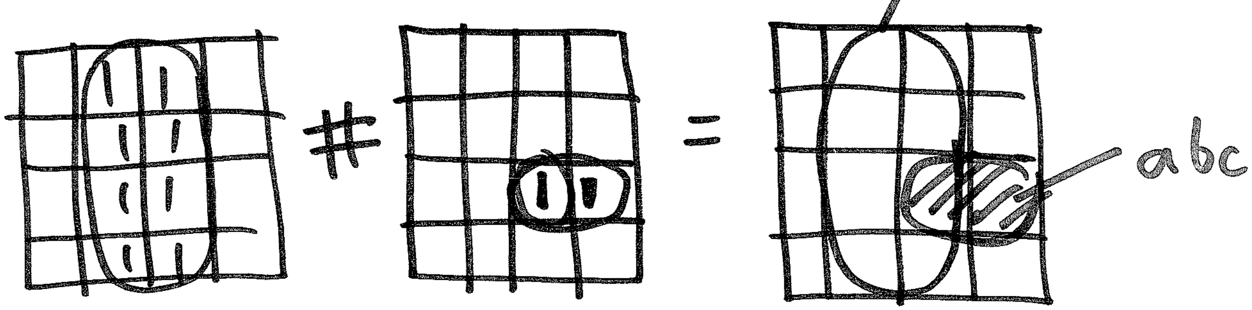
F

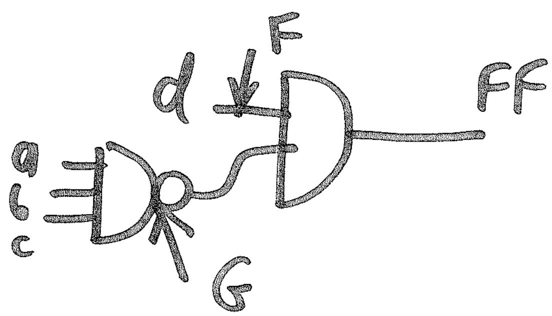
G

FF

$$F \# G = F \cdot \overline{G}$$







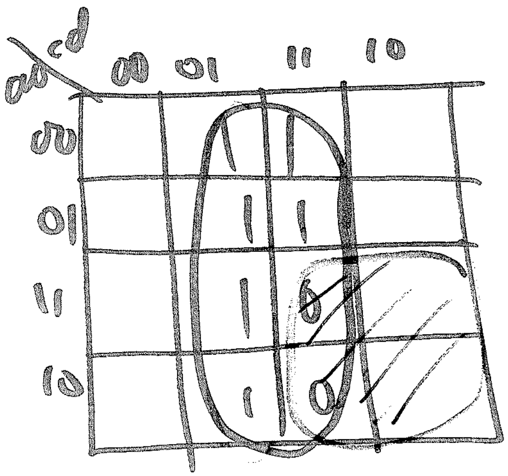


Illustration of graphical subtraction

$d \cdot \overline{ac}$

$= d \cdot$

$d \# ac$

↑ sharp, cube calculus operation

Example of multi-output design Using inhibition

0	0	1	1
0	0	0	1
1	1	0	0
1	1	0	1

f_1

1			
		1	
		1	1
		X	

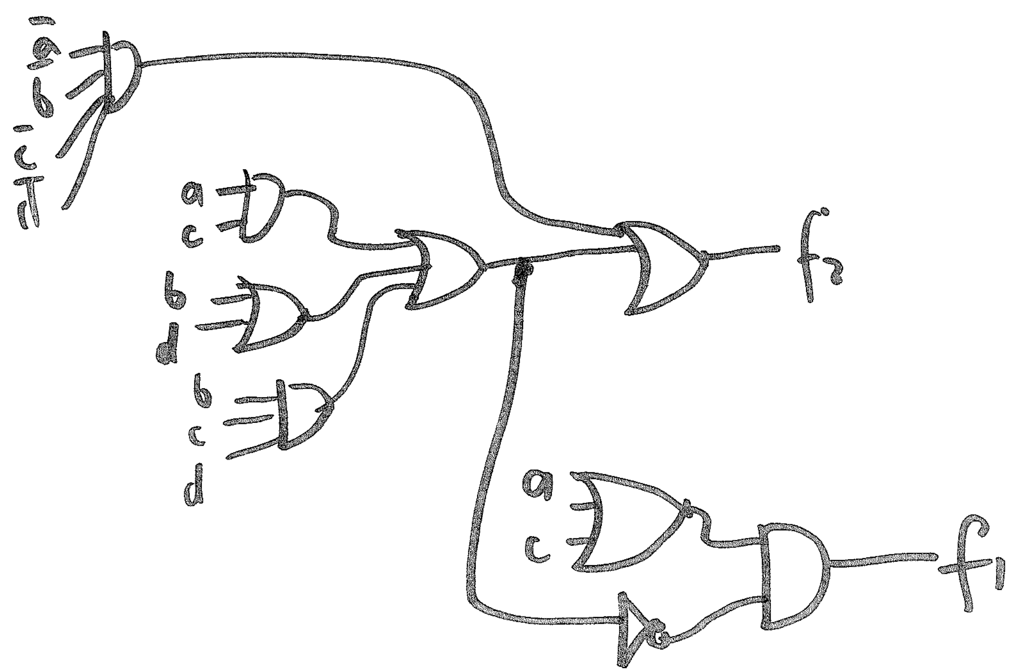
f_2

bcd

abc

acd

$ac(b+d)+bcd$

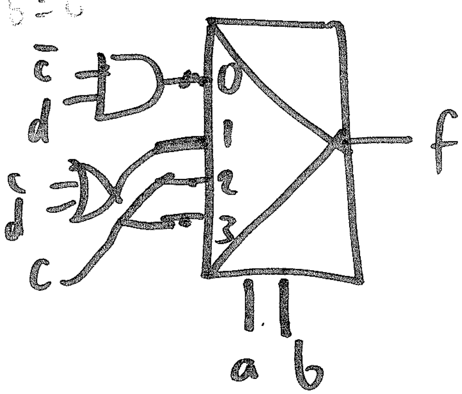


(IV) Using multiplexers in synthesis

ab \ cd	00	01	11	10
00	0	1	0	0
01	0	1	1	1
11	0	0	1	1
10	0	0	1	1

f

a=0
b=0



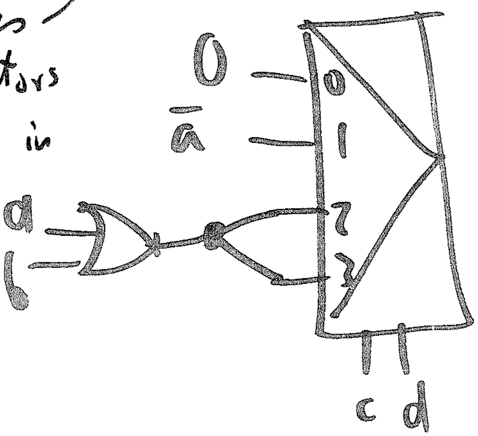
a=0 b=1

cd	00	01	11	10
	0	1	1	1

a=1 b=0

ab \ cd	00	01	11	10
00	0	1	0	0
01	0	1	1	1
11	0	0	1	1
10	0	0	1	1

shapes of cofactors for c, d in address



ab \ cd	00	01	11	10
00	0	1	0	0
01	0	1	1	1
11	0	0	1	1
10	0	0	1	1

a c

b \ d	0	1
0	0	1
1	0	1

shapes of cofactors for variables a, c