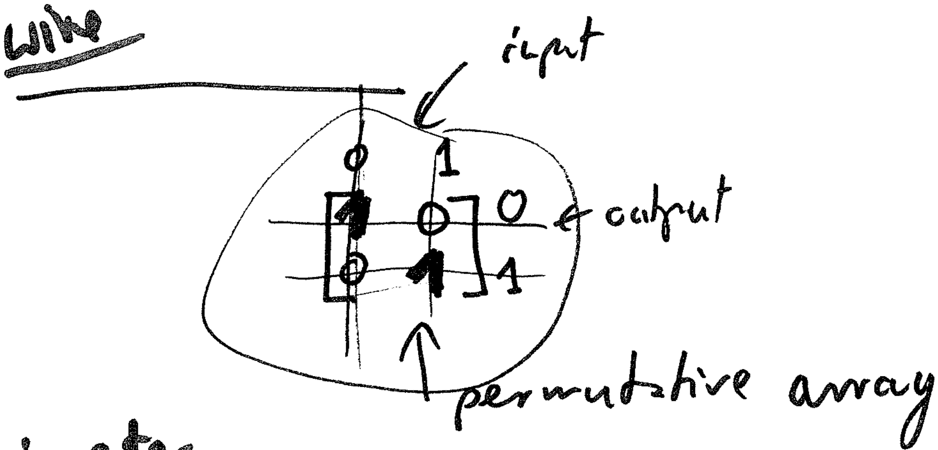
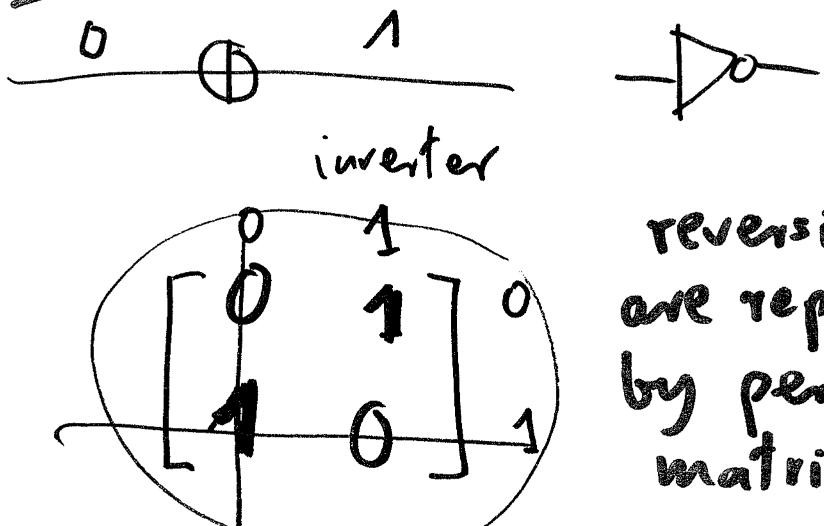
Reversible gates (lecture 3)

1) wire

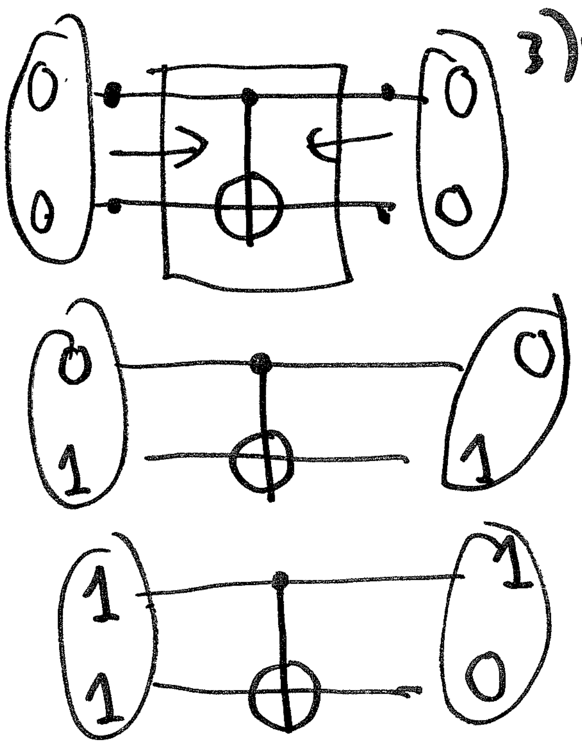


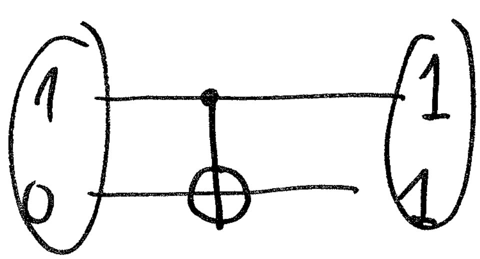
2) inverter



reversible gates are represented by permutative matrices

3) Feynman gate





$00 \to 00$
$01 \to 01$
$11 \to 10$
$10 \to 11$

4) Unitary matrix for Feynman ②

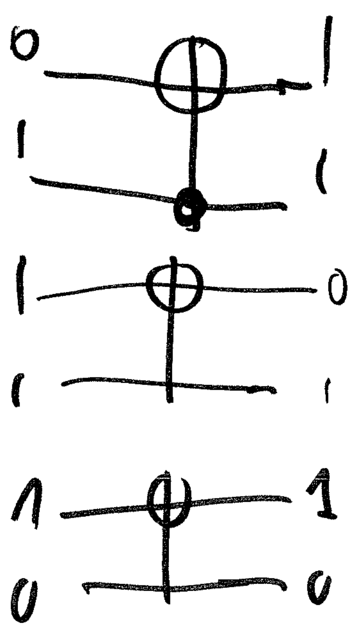
$$
\begin{array}{c|c}
 & \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\
\hline
\begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array} &
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}
\end{array}
$$

this is quantum gate.

If inputs only binary, then reversible.

It is also permutative!

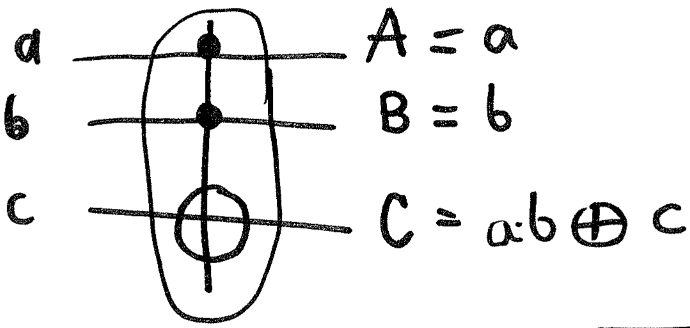
Variant of Feynman



$$
\begin{array}{c|c}
 & \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\
\hline
\begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array} &
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}
\end{array}
$$

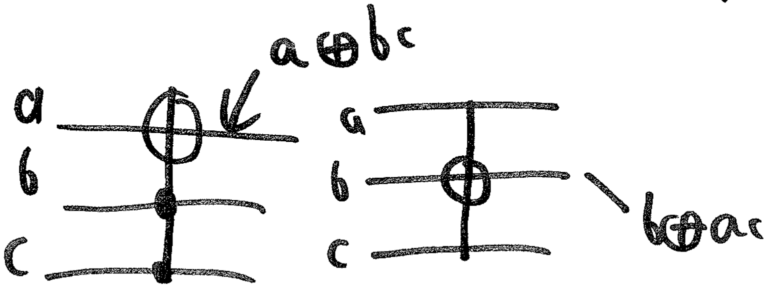
exor up Feynman has also permutative matrix!

5) Toffoli Gate



$$ab \oplus 1 = \overline{ab} = \bar{a} + \bar{b}$$

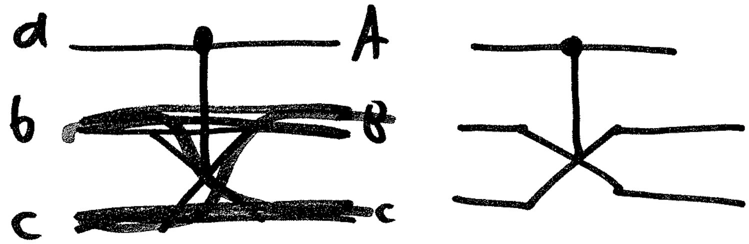
$$
\begin{array}{c|cccccccc|c}
abc & 000 & 001 & 010 & 011 & 100 & 101 & \overbrace{110 \quad 111}^{a=b=1} & & \\
\hline
 & 1 & 0 & 0 & 0 & & & & & 000 \\
 & 0 & 1 & 0 & 0 & & & 0 & & 001 \\
 & 0 & 0 & 1 & 0 & & & & & 010 \\
 & 0 & 0 & 0 & 1 & & & & & 011 \\
 & & & & & 1 & 0 & & & 100 \\
 & & 0 & & & 0 & 1 & & & 101 \\
 & & & & & & & 0 & 1 & 110 \\
 & & & & & & & 1 & 0 & 111
\end{array}
$$



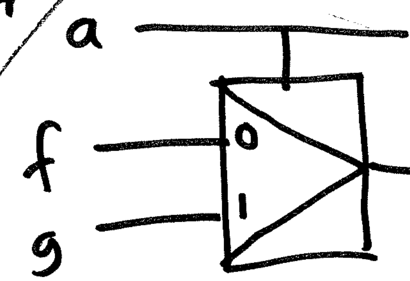
a=1
b=1

All Toffoli gates, for any # of inputs, are described by permutative matrices

6) Fredkin

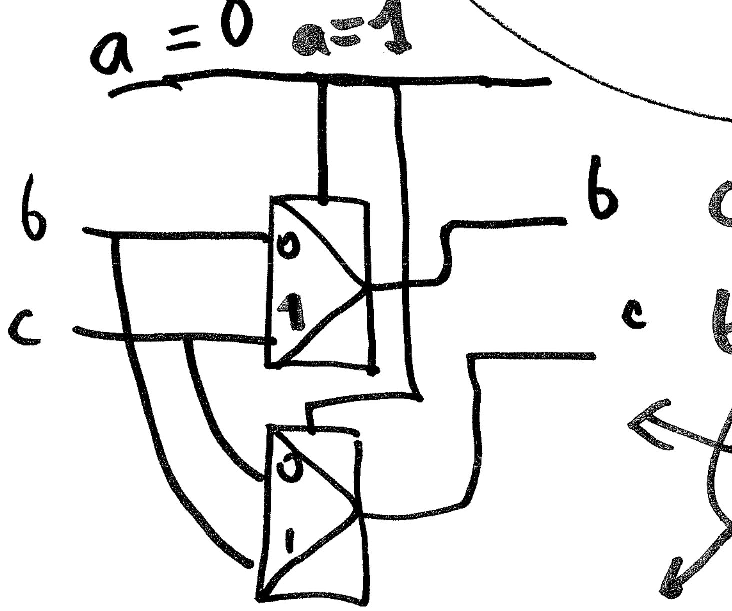


multiplexer

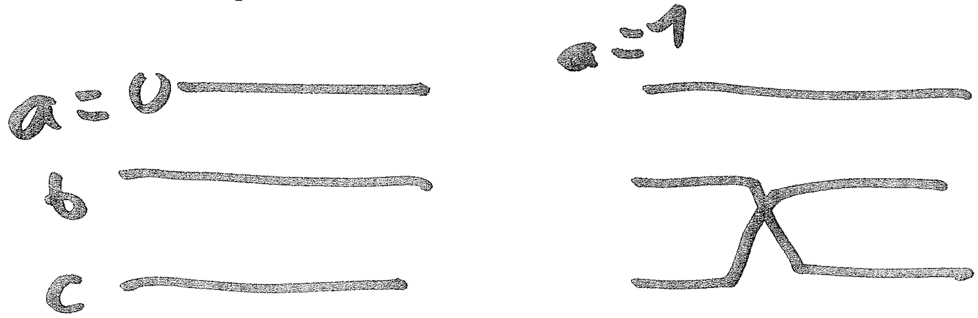


$$F = \begin{cases} f & \text{when } a = 0 \\ g & \text{when } a = 1 \end{cases}$$

Find the matrix – do it!

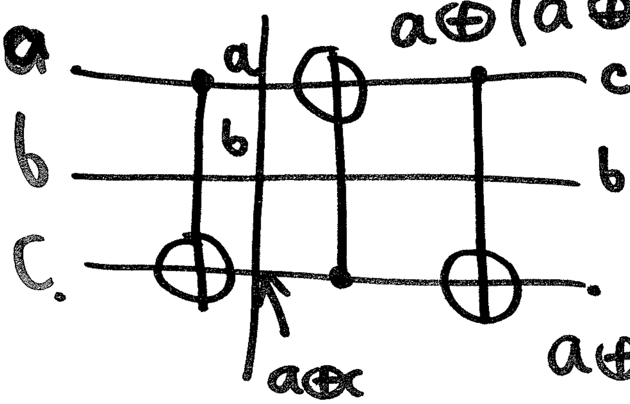


Fredkin is controlled-swap gate



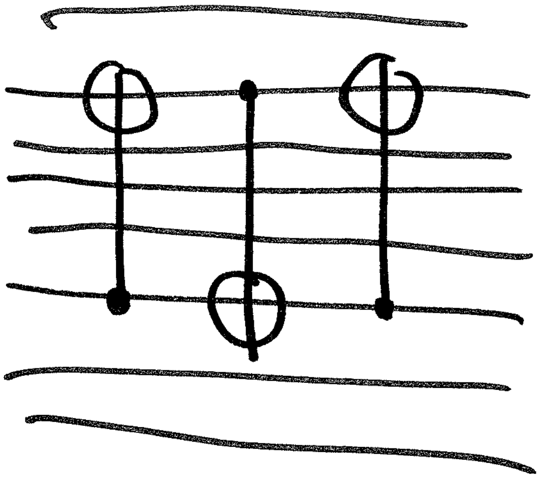
controlled-swap

$a \oplus (a \oplus c) = c$



swapped a and c

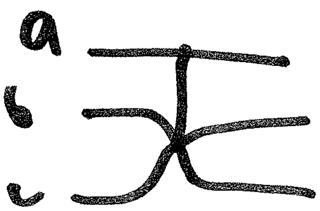
$a \oplus c \oplus c = a$



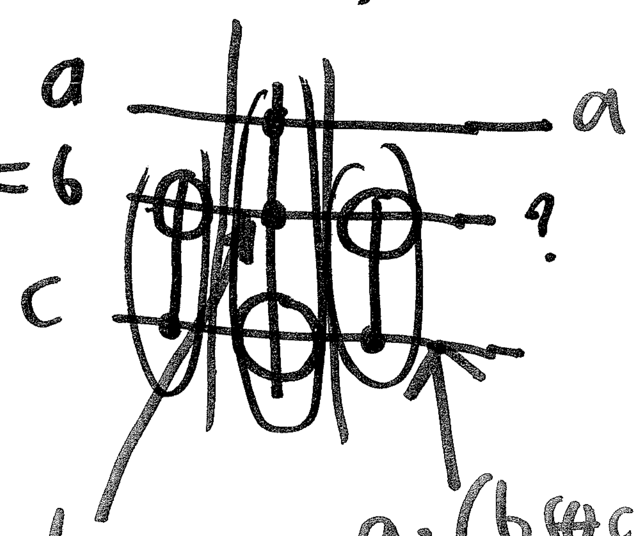
analyze

SWAP GATE FROM Feynman gates

How to realize Fredkin from Toffoli and Feynman?



=



$b \oplus c$

$\underline{a \cdot (b \oplus c)} \oplus c$

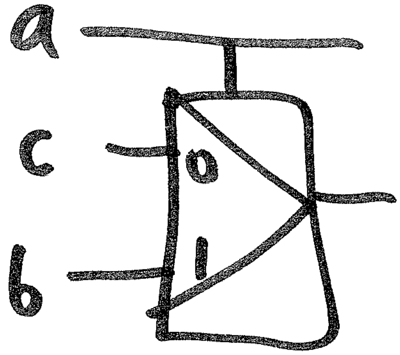
$= ab \oplus ac \oplus c$

$= ab \oplus c(a \oplus 1)$

$$= ab \oplus c(a \oplus 1)$$

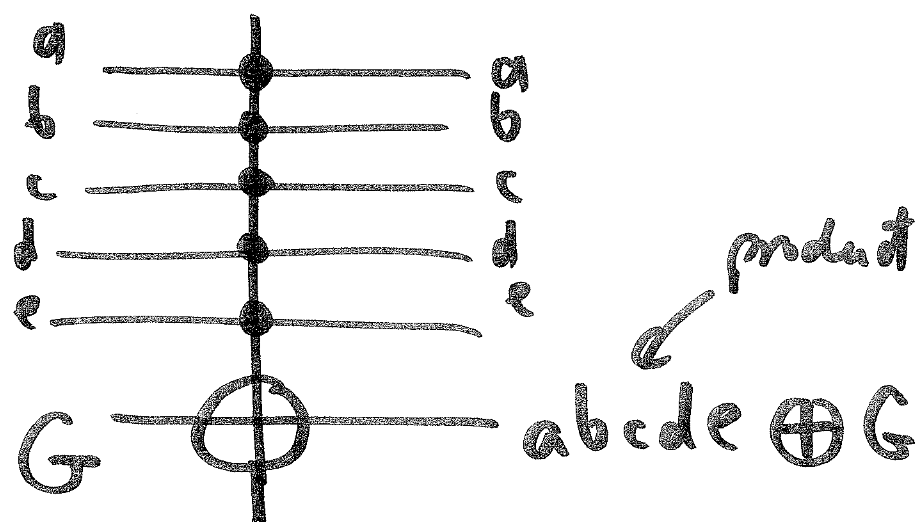
$$= ab \oplus c\,\bar{a}$$

$$= ab + \bar{a}c$$

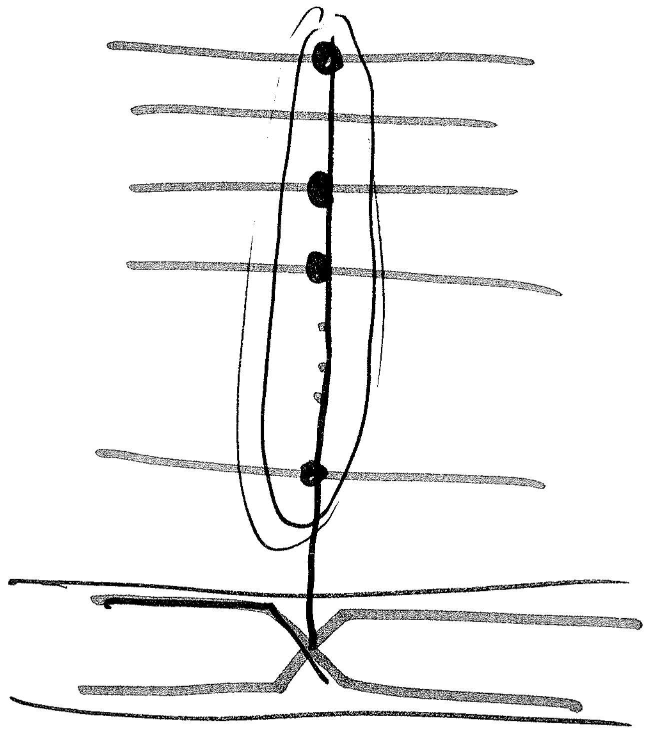


Conclusion

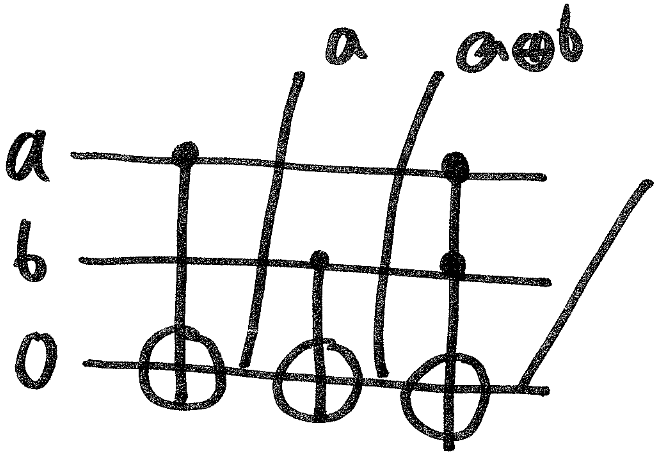
We will use gates NOT, Feynman, Toffoli and Fredkin



Toffoli with many inputs



Fredkin controlled by big product



$$a+b = a \oplus b \oplus ab$$

realization of OR is expensive.

We do not use OR gates

- expensive
- lead to circuits with many ancilla bits

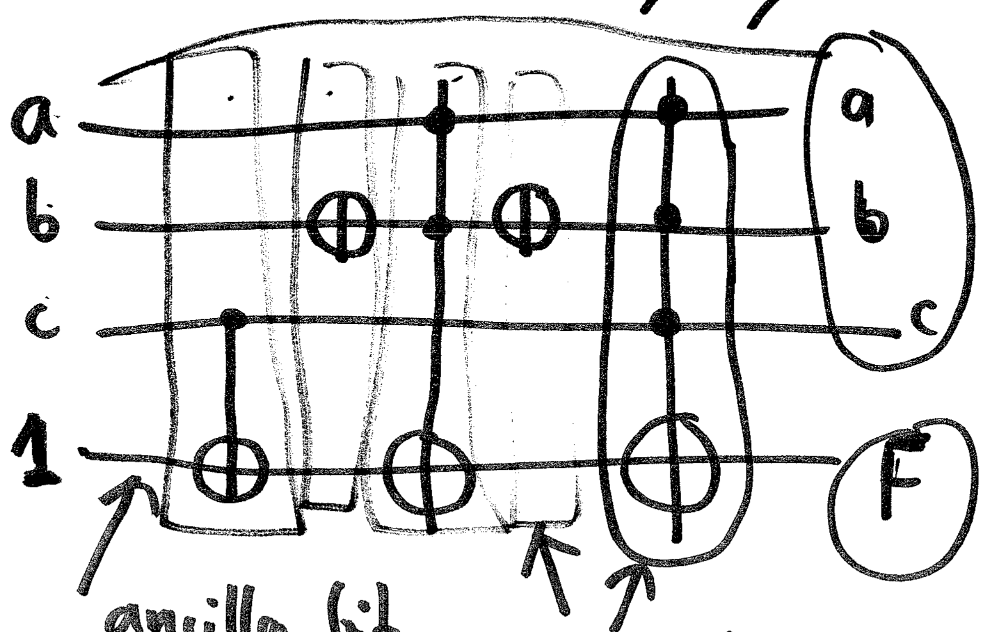
Realization of ESOP as reversible circuit

Theorem

Every ESOP is realized using NOT, Fe, Toffoli with one ancilla bit.

All inputs go through.

$$F = a\bar{b} \oplus abc \oplus \cancel{c} \oplus \cancel{1}$$



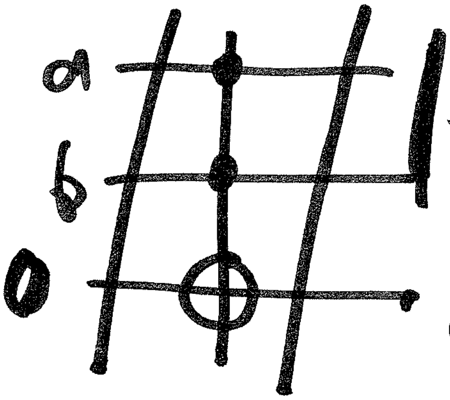
ancilla bit set to 1

reversible gates.

$\boxed{\text{ESOP is composition of reversible gates}}$

8A

$f = ab$



$ab \oplus 0 = ab$

Remark 1

Our method automatically converts functions from non-reversible to reversible.

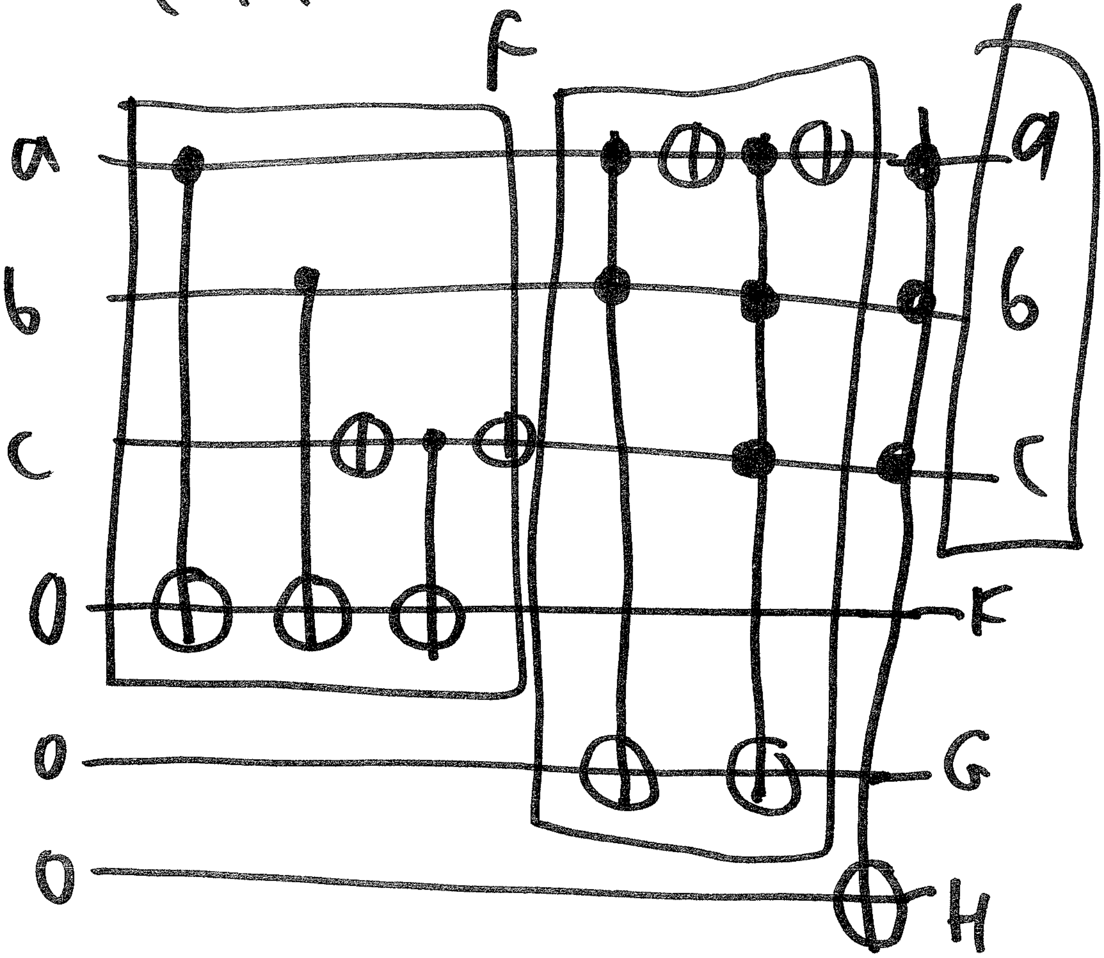
Remark 2

Single product requires one ancilla bit set to 0 as shown above

$F(a,b,c) = \cancel{a} \oplus \cancel{b} \oplus \overset{\bar{c}}{c \oplus 1}$

$G(a,b,c) = ab \oplus \bar{a}bc$

$H(a,b,c) = abc$



quantum array

Here we realized 3-output function (F, G, H) with 3 ancilla bits set to 0

Analysis of reversible circuits.

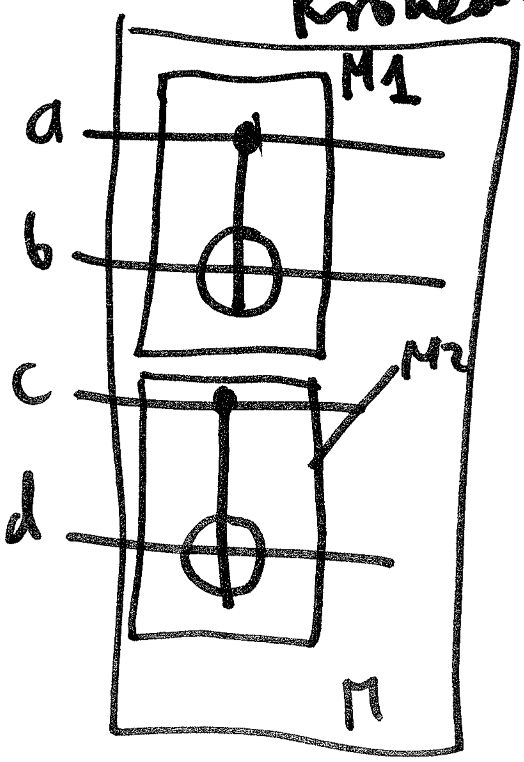
If two gates are connected in parallel, to calculate their permutative matrix we do a Kronecker product of matrices of these gates.

Kronecker product
= tensor product
= tensor multiplication.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{bmatrix} x & y \\ z & v \end{bmatrix}$$

$$= \begin{bmatrix} a\begin{bmatrix} x & y \\ z & v \end{bmatrix} & b\begin{bmatrix} x & y \\ z & v \end{bmatrix} \\ c\begin{bmatrix} x & y \\ z & v \end{bmatrix} & d\begin{bmatrix} x & y \\ z & v \end{bmatrix} \end{bmatrix} = \begin{bmatrix} ax & ay & bx & by \\ az & av & bz & bv \\ cx & cy & dx & dy \\ cz & cv & dz & dv \end{bmatrix}$$

Kronecker Product



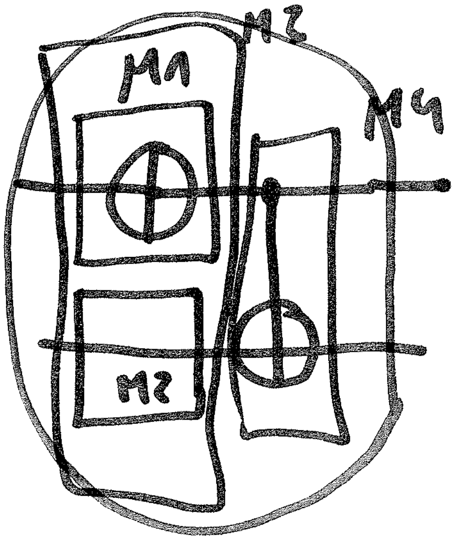
$$M = M1 \otimes M2$$

$$\begin{array}{c} ab \\ \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array} \qquad M1 \quad M2$$

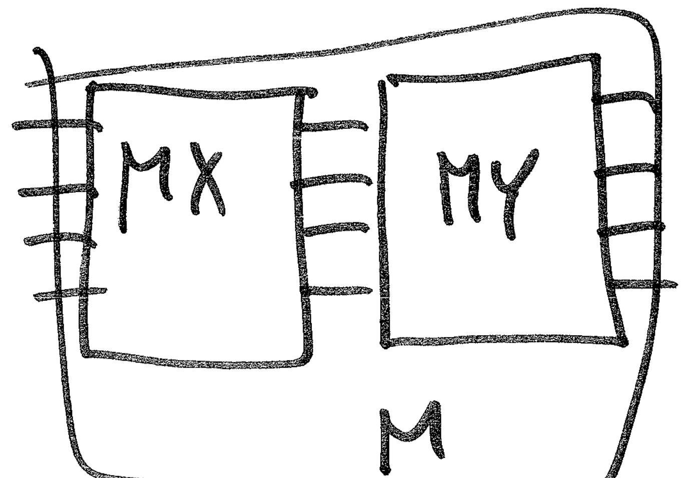
$$M_1 \otimes M_2 \begin{bmatrix} 1\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & & 1 \\ & & 1 & \end{bmatrix} & 0\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & & 1 \\ & & 1 & \end{bmatrix} & 0\begin{bmatrix} & \end{bmatrix} & 0\begin{bmatrix} & \end{bmatrix} \\ 0\begin{bmatrix} & \end{bmatrix} & 1\begin{bmatrix} & \end{bmatrix} & 0\begin{bmatrix} & \end{bmatrix} & 0\begin{bmatrix} & \end{bmatrix} \\ 0\begin{bmatrix} & \end{bmatrix} & 0\begin{bmatrix} & \end{bmatrix} & 0\begin{bmatrix} & \end{bmatrix} & 1\begin{bmatrix} & \end{bmatrix} \\ 0\begin{bmatrix} & \end{bmatrix} & 0\begin{bmatrix} & \end{bmatrix} & 1\begin{bmatrix} & \end{bmatrix} & 0\begin{bmatrix} & \end{bmatrix} \end{bmatrix}$$

$M_1 \otimes M_2$

$$
\left[\begin{array}{c|c|c|c}
\begin{matrix} 1 & & & \\ & 1 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{matrix} & 0 & 0 & 0 \\ \hline
0 & \begin{matrix} 1 & & & \\ & 1 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{matrix} & 0 & 0 \\ \hline
0 & 0 & 0 & \begin{matrix} 1 & & & \\ & 1 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{matrix} \\ \hline
0 & 0 & \begin{matrix} 1 & & & \\ & 1 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{matrix} & 0
\end{array}\right]
$$



The permutive matrix of serial connection of gates with matrice MX and MY is MY * MX



M= MY * MX

Please pay attention to reverse order of matrices.

Prove why it is so?

14 A

$$M1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \leftarrow \text{inverter}$$

$$M2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \leftarrow \text{wire}$$

$$M_1 \otimes M_2 = \begin{bmatrix} 0 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 1 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ 1 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix}$$

Properties of Kronecker product

$$\boxed{a \otimes (b \otimes c) = (a \otimes b) \otimes c}$$

$$\cancel{a \times b = b \times a}$$

example continued

$$M_1 \otimes M_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$M_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\cancel{M = M_1 \otimes M_2}$

$$M = M_4 * (M_1 \otimes M_2)$$

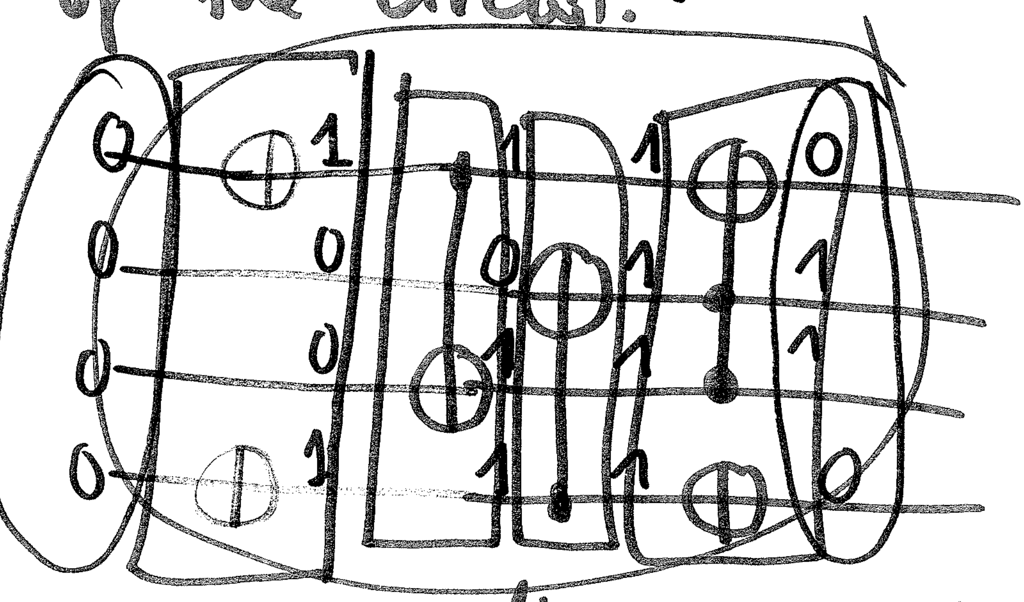
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} * \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ & \ddots & & \\ & & \ddots & \\ & & & \end{bmatrix}$$

HOMEWORK 2

PART 1

Given is the following circuit.

1. Use Matlab to calculate all intermediate Matrices and the final matrix of the circuit. M



suggestion — use swap gates.

(17)

1. Given matrix M calculate the output state for the input state 0000

$$0000 = \begin{array}{l} 0 \begin{bmatrix}1\\0\end{bmatrix} \text{———} \\ 0 \begin{bmatrix}1\\0\end{bmatrix} \text{———} \\ 0 \begin{bmatrix}1\\0\end{bmatrix} \text{———} \\ 0 \begin{bmatrix}1\\0\end{bmatrix} \text{———} \end{array}$$

$|0\rangle$

$$0 \otimes 0 \otimes 0 \otimes 0$$

$$= \begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix} = \left.\begin{bmatrix}1\\0\\0\\ \vdots \\ 0\end{bmatrix}\right\} 16$$

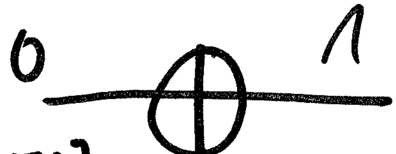
$$\begin{bmatrix} 1\begin{bmatrix}1\\0\end{bmatrix} \\ 0\begin{bmatrix}1\\0\end{bmatrix} \end{bmatrix} = \begin{array}{c|c} \begin{bmatrix}1\\0\\0\\0\end{bmatrix} & \begin{array}{c}00\\01\\10\\11\end{array} \end{array} \qquad 8 \quad \overset{000}{\begin{bmatrix}1\\0\\ \vdots \\ 0\end{bmatrix}}$$

$$16 \begin{bmatrix}1\\0\\0\\ \vdots \end{bmatrix}$$

$$0 = \begin{bmatrix}1\\0\end{bmatrix}$$

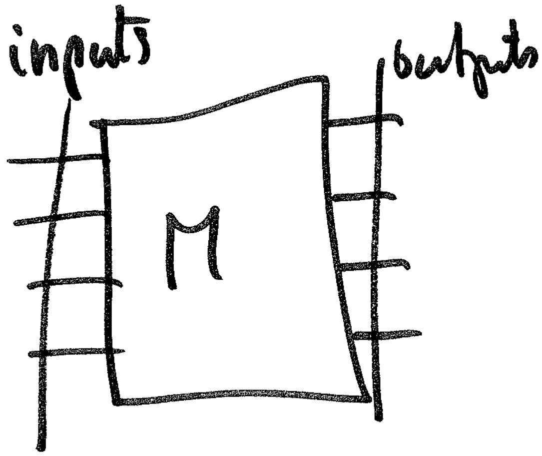
$$1 = \begin{bmatrix}0\\1\end{bmatrix}$$

$$\begin{bmatrix}\langle\text{placeholder for signal zero}\rangle\\ \langle\text{placeholder for signal one}\rangle\end{bmatrix}$$



$$\begin{bmatrix}1\\0\end{bmatrix}$$

$$\begin{bmatrix}0&1\\1&0\end{bmatrix}\times\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}0+0\\1\cdot1+0\cdot0\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}$$



$$\begin{bmatrix}16\times16\end{bmatrix}\begin{bmatrix}16\\\times1\end{bmatrix}$$

outputs = M * inputs

$M = M1 \otimes M2$

```
>> M = kron(M1, M2)
```

in Matlab

```
M1 = [0 1 ; 1 0]
```

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

```
>>
```

M1 * M2

All these are easy
in Matlab

Positive Polarity Reed Muller

We start from DSOP

DSOP$_1$

$ab \backslash cd$	00	01	11	10
00	0	0	0	0
01	0	0	1	0
11	0	1	1	1
10	0	1	1	1

bcd

$ac\bar{d}$

$a\bar{c}d$

$a\bar{b}cd$

$$DSOP = bcd \oplus ac\bar{d} \oplus a\bar{c}d \oplus a\bar{b}cd$$

$$= \cancel{bcd} \oplus \cancel{ac}(d \oplus 1) \oplus a(\cancel{c} \oplus 1)d \oplus a(\cancel{b \oplus 1})cd = bcd \oplus \cancel{acd} \oplus ac \oplus \cancel{acd} \oplus ad \oplus abcd \oplus acd$$

DSOP$_2$

		1	
	1	1	1
	1	1	1

$\bar{a}bcd$

ad

$ac\bar{d}$

$$f = ad \oplus ac(1 \oplus d) \oplus (1 \oplus a)bcd$$

$$= ad \oplus ac \oplus acd \oplus bcd \oplus abcd$$

I calculated PPRM form from $DSOP_1$ and $DSOP_2$.

Because function was the same in both cases I obtained the same PPRM form in both cases equal

$abcd \oplus acd \oplus ad \oplus ac \oplus bcd$

positive polarity.

Part 3 of homework

a) Given is function G

ab \ c	0	1
00	0	1
01	0	1
11	1	0
10	0	1

Find the PPRM of this function.

b) Find all FPRMs of this function.

end of homework for today

we will cont on Wed.

PPRM

$$f = \underline{a} \oplus \underline{b} \oplus \underline{cde} \oplus \underline{ac}$$

a b c d e
1 1 1 1 1

$$f = \bar{a} \oplus b \oplus \bar{a}b \oplus \bar{a}\,\bar{c} \oplus b\bar{c}$$

a b c
0 1 0 $\Leftarrow$ polarity.

(23)

$\bar{a}\bar{c}$ polarity 00

10 Pola

$= (1 \oplus a)\,\bar{c} = \bar{c} \oplus a\bar{c}$

Pola 11

$= (1 \oplus a)(1 \oplus c)$

$= 1 \oplus a \oplus c \oplus ac$

FOR FUNCTION OF n variables there is 2^n various polarities.

there is 2^n FPRM forms.

We select the best form, with minimum cost.