

From Fourier Expansions to Arithmetic-Haar Expressions on Quaternion Groups

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Abstract. Arithmetic expressions for switching functions are introduced through the replacement of Boolean operations with arithmetic equivalents. In this setting, they can be regarded as the integer counterpart of Reed-Muller expressions for switching functions. However, arithmetic expressions can be interpreted as series expansions in the space of complex valued functions on finite dyadic groups in terms of a particular set of basic functions. In this case, arithmetic expressions can be derived from the Walsh series expansions, which are the Fourier expansions on finite dyadic groups.

In this paper, we extend the arithmetic expressions to non-Abelian groups by the example of quaternion groups. Similar to the case of finite dyadic groups, the arithmetic expressions on quaternion groups are derived from the Fourier expansions. Attempts are done to get the related transform matrices with a structure similar to that of the Haar transform matrices, which ensures efficiency of computation of arithmetic coefficients.

Keywords: Fourier transform, Quaternion groups, Arithmetic expressions, Haar expressions.

1 Introduction

Fourier analysis on finite non-Abelian groups is an alternative to spectral techniques on finite Abelian groups, offering some advantages in practical engineering applications, see for example, [10], [12], [13], [14]. The applications of Fourier transform on non-Abelian groups in switching theory and logic design are suggested in [9], and [11].

In a review of theory and applications of Fourier analysis on finite groups in engineering practice [34], the following problem was formulated

“The ultimate purpose must be to find out whether this group (the quaternion Q_8) may be as significant for logic synthesis as it seems to be for filtering and other signal processing tasks”.

In [28], [29], it is compared the space and time complexity of FFT for switching functions of a given number of variables n considered as functions on C_2^n , where $C_2 = \{0, 1\}$, \oplus , where \oplus is the addition modulo 2 usually denoted as EXOR, and the quaternion groups $Q_{8^{n/3}}$, $C_2 Q_8^{(n-1)/3}$, and $C_4 Q_8^{(n-2)/3}$. It is shown that the quaternion groups provide a considerable reduction of time complexity at the price of relatively small increase of the space complexity. For example, if $n \geq 14$, the FFT on $C_4 Q_8^4$ can be performed 10 times faster than on C_2^{14} , with a three times increased space.

In [26], [29], it is shown that the use of non-Abelian groups, in particular the quaternion groups, as the domain groups for decision diagrams representations of switching and multiple-output switching functions permits reduction of the complexity of their DD representations.

This paper is a continuation of the research towards a more complete answer to the question of efficiency of applications of the quaternion groups in switching theory and logic design. We introduce arithmetic expressions on the quaternion groups corresponding to the arithmetic expressions on C_2^n . The interest in arithmetic expressions on Q_8^r is motivated by the renewed recent interest in arithmetic expressions on C_2^n originating in their properties and applications

1. Parallelization of algorithms for large switching functions [15], [16],
2. Single expression for multi-output switching functions [20],
3. Some word-level DD are graphic representations of arithmetic expressions for discrete functions [25], [27], [31],
4. Different applications of arithmetic expressions such, as for example, checking of error probability in logic networks [15], [35].

In this paper, we extended a method for derivation of polynomial expressions from Walsh (Fourier) expansions on C_2^n to non-Abelian groups by using as the example the groups Q_8 . For other values of n , we use the groups $Q_8^{n/3}$, $C_2 Q_8^{(n-1)/3}$, and $C_4 Q_8^{(n-2)/3}$. In this case, the arithmetic expressions are generated by the Kronecker product of the arithmetic matrices on C_2 and Q_8 . We consider groups of orders 2^n , therefore, it is possible to derive arithmetic expressions on quaternion groups in terms of switching variables. Extension to groups of arbitrary orders is possible by allowing variables to take values in arbitrary finite sets. An example of such generalizations for p -adic groups, for example $p = 3$, is given in [18].

We organize the presentation in this paper as follows. In Section 2, we introduce some basic notations and definitions. The subject of Section 3 is the definition of the arithmetic expressions in $C(C_2^n)$. In Section 4, we discuss the

derivation of arithmetic expressions from Walsh (Fourier) expansions on finite dyadic groups. We present a generalization of this method to non-Abelian groups through the example of quaternion groups in Section 5. Furthermore, we define arithmetic-Haar expressions. In Section 6, we define fixed-polarity Fourier expansions on quaternion groups and fixed-polarity arithmetic-Haar expressions. In Section 7, we consider calculation of arithmetic-Haar coefficients through FFT and decision diagrams. In Section 8, we give some closing remarks and suggestions for future work.

2 Notations and Definitions

2.1 Discrete Functions

Denote by f a function $f : G \rightarrow P$, where G is a finite Abelian group of order g , and P is a field that may be a finite (Galois) field, the field of rational numbers Q , or the complex field C . If G is decomposable into the product of n subgroups G_i of orders g_i ,

$$G = \times_{i=1}^n G_i, \quad g = \prod_{i=1}^n g_i, \quad (1)$$

then $f(x)$, $x \in G$ can be alternatively considered as an n variable function $f(x_1, \dots, x_n)$, $x_i \in G_i$.

The set of all such functions under usual addition of functions and the multiplication by a scalar $\alpha \in P$ endorse the structure of a linear vector space $P(G)$.

A discrete function defined in g points of a group G can be considered as a function of different number of variables depending on the decomposition assumed for G .

Example 1 A switching function of three variables $f(x_1, x_2, x_3)$, $x_i \in \{0, 1\}$, can be regarded as a function on the finite dyadic group of order 8, and represented by a vector $\mathbf{F} = [f(000), f(001), \dots, f(111)]^T$. Alternatively, f can be considered as a function on the quaternion group Q_8 of order 8 defined in what follows. Thus, it is given by a vector $\mathbf{F} = [f(0), f(1), \dots, f(7)]^T$. Values for f are considered as logic values 0 and 1. In this case, they are taken in $GF(2)$. Alternatively, values for f can be considered as integers 0 and 1, a subset of the complex-field C .

2.2 Fourier Expressions on Finite Groups

Elements of group representations $R_w^{(i,j)}$ of a group G over a field P form an orthogonal complete set, thus, basis in $P(G)$ [8], [33]. The Fourier transform in $\mathcal{F}(G)$ is defined in terms of this basis.

Definition 1 *The Fourier transform on a finite non-Abelian group G of order g is defined as a mapping $f(x) \rightarrow S_f(w), x \in G, w \in \Gamma$, where Γ is the dual object for G . This mapping is given by*

$$\mathbf{S}_f(w) = r_w g^{-1} \sum_{u=0}^{g-1} f(u) \mathbf{R}_w(u^{-1}),$$

$$f(x) = \sum_{w=0}^{K-1} Tr(\mathbf{S}_f(w) \mathbf{R}_w(x)),$$

where $Tr(\mathbf{Q})$ is the trace of \mathbf{Q} , g is the order of G , K is the order of Γ , \mathbf{R}_w denotes the unitary irreducible group representations, and r_w is the order of \mathbf{R}_w .

In matrix notation,

$$[\mathbf{S}_f] = g^{-1} [\mathbf{R}]^{-1} \odot \mathbf{F},$$

$$\mathbf{F} = [\mathbf{R}] \circ [\mathbf{S}_f],$$

where \odot and \circ are generalized matrix multiplications permitting multiplication of rectangular matrices whose entries are matrices [24], [30], [33].

If G is a non-Abelian group, then at least one $r_w > 1$, and it holds $\sum_{w \in \Gamma} r_w^2 = g$. If G is an Abelian group, then all the unitary irreducible representations are one-dimensional, thus, reduce to the group characters $\chi_w(x)$, $w, x \in G$. In this case, Γ has the structure of a multiplicative group isomorphic to G .

2.3 Finite Dyadic Groups

Group representations for C_2 are given by the basic Walsh matrix

$$\mathbf{W}(1) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

The group representations of the finite dyadic group C_2^n are given by the Walsh matrix of order n defined as

$$\mathbf{W}(n) = \bigotimes_{i=1}^n \mathbf{W}(1),$$

where \otimes denotes the Kronecker product.

Assume that $f \in C(C_2^n)$ is given by the vector $\mathbf{F} = [f(0), \dots, f(2^n - 1)]^T$. If the Walsh (Fourier) spectrum for f is represented by a vector $\mathbf{S}_{w,f} = [S_{w,f}(0), \dots, S_{w,f}(2^n - 1)]^T$, then

$$\mathbf{S}_{w,f} = 2^{-n} \mathbf{W}(n) \mathbf{F},$$

and

$$\mathbf{F} = \mathbf{W}(n)\mathbf{S}_{w,f},$$

since the Walsh matrix is a self-inverse matrix over C with the scaling factor 2^{-n} .

Example 2 The Walsh spectrum of a three-variable switching function $f(x_1, x_2, x_3)$ given by the truth-vector $\mathbf{F} = [1, 0, 0, 0, 0, 1, 1, 1]^T$ is given by $\mathbf{S}_{w,f} = [4, 0, 0, 0, -2, 2, 2, 2]^T$. Thus,

$$f(x) = \frac{1}{8}(4w_0(x) - 2w_4(x) + 2w_5(x) + 2w_6(x) + 2w_7(x)), \quad (2)$$

where $w_i(x)$ represent the columns of the corresponding Walsh matrix (see Section 5).

2.4 Quaternion Groups

The quaternion group Q_8 is generated by two elements a and b , say, and the group identity is denoted by e . If the group operation is written as abstract multiplication, the following relations hold for the group generators: $b^2 = a^2, bab^{-1} = a^{-1}, a^4 = e$. All the irreducible unitary representations of Q_8 over C are given in Table 1.

The Fourier transform on Q_8 is defined in terms of the group representations given by the columns of the matrix whose entries are determined from Table 1. Thus,

$$\mathbf{Q}_8 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & -1 & -1 & i & 0 & 0 & -i \\ 1 & 1 & 1 & 1 & -1 & 0 & 0 & -1 \\ 1 & 1 & -1 & -1 & -i & 0 & 0 & i \\ 1 & -1 & 1 & -1 & 0 & -1 & 1 & 0 \\ 1 & -1 & -1 & 1 & 0 & -i & -i & 0 \\ 1 & -1 & 1 & -1 & 0 & 1 & -1 & 0 \\ 1 & -1 & -1 & 1 & 0 & i & i & 0 \end{bmatrix}.$$

The Fourier transform matrix is given by the matrix inverse to \mathbf{Q}_8 . Thus,

$$\mathbf{Q}_8^{-1} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 2 & -2i & -2 & 2i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 2i & 2 & -2i \\ 0 & 0 & 0 & 0 & 2 & 2i & -2 & -2i \\ 2 & 2i & -2 & -2i & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Table 1. Unitary irreducible representations of Q_8 over C

x	R_0	R_1	R_2	R_3	R_4
0	1	1	1	1	I
1	1	1	-1	-1	$i\mathbf{A}$
2	1	1	1	1	$-\mathbf{I}$
3	1	1	-1	-1	$i\mathbf{B}$
4	1	-1	1	-1	C
5	1	-1	-1	1	$-i\mathbf{D}$
6	1	-1	1	-1	E
7	1	-1	-1	1	$i\mathbf{D}$
	$r_0 = 1$	$r_1 = 1$	$r_2 = 1$	$r_3 = 1$	$r_4 = 2$
	$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$			
	$\mathbf{B} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$	$\mathbf{C} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$			
	$\mathbf{D} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\mathbf{E} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$			

We denoted by $C(Q_8)$ the space of functions $f : Q_8 \rightarrow C$. For $f \in C(Q_8)$ given by the vector $\mathbf{F} = [f(0), f(1), f(2), f(3), f(4), f(5), f(6), f(7)]^T$, the Fourier spectrum given by the vector $\mathbf{S}_{q,f} = [S_{q,f}(0), S_{q,f}(1), S_{q,f}(2), S_{q,f}(3), S_{q,f}(4), S_{q,f}(5), S_{q,f}(6), S_{q,f}(7)]^T$ is determined as

$$\mathbf{S}_{q,f} = \frac{1}{8}\mathbf{Q}_8^{-1}\mathbf{F},$$

and

$$\mathbf{F} = \mathbf{Q}_8\mathbf{S}_{q,f}.$$

Example 3 If f in Example 2 is considered as a function on Q_8 , then the Fourier spectrum for f is given by $\mathbf{S}_{q,f} = \frac{1}{8}[4, -2, 0, 2, 2, 2, -2, 2]^T$. Thus,

$$f(x) = \frac{1}{8}(4q_0(x) - 2q_1(x) + 2q_3(x) + 2q_4(x) + 2q_5(x) - 2q_6(x) + 2q_7(x)),$$

where $q_i(x)$ are columns of the matrix \mathbf{Q}_8 .

3 Arithmetic Expressions

In the matrix notation, the arithmetic expressions for functions in $C(C_2^n)$ are defined as

$$\begin{aligned}
 f &= \mathbf{X}_a(n)\mathbf{S}_{a,f} \\
 &= \left(\bigotimes_{i=1}^n [1 \ x_i] \right) \mathbf{S}_{a,f}
 \end{aligned}$$

with

$$\mathbf{S}_{a,f} = \mathbf{A}(n)\mathbf{F},$$

$\mathbf{F} = [f(0), \dots, f(2^n - 1)]^T$, and

$$\mathbf{A}(n) = \bigotimes_{i=1}^n \mathbf{A}(1),$$

where $\mathbf{A}(1)$ is the basic arithmetic transform matrix given by

$$\mathbf{A}_f(1) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

If the elements of \mathbf{X}_a are interpreted as logic values 0 and 1, then this matrix is referred to as the Reed-Muller matrix or the conjunctive transform matrix [2], [3]. It defines a self-inverse transform in $GF_2(C_2^n)$ denoted as the Reed-Muller transform, or the conjunctive transform. In this context, the arithmetic transform in $C(C_2^n)$ is denoted as the inverse conjunctive transform [2]. For more details on arithmetic expressions, see for example, [2], [15], [16], [21], [35].

Example 4 For $n = 3$, the arithmetic transform in $C(C_2^3)$ is defined by the matrix

$$\mathbf{A}(3) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 \\ -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \end{bmatrix}.$$

For f in Example 2, the arithmetic spectrum is given by $\mathbf{A}_f = [1, -1, -1, 1, -1, 2, 2, -2]^T$. Therefore,

$$f = 1 - x_3 - x_2 + x_2x_3 - x_1 + 2x_1x_3 + 2x_1x_2 - 2x_1x_2x_3.$$

4 Arithmetic Expressions from Walsh Expansions

Let $G = C_2^2$ and $P = C$. Each $x \in G$ can be expressed by $x = (x_1, x_2)$, $x_1, x_2 \in \{0, 1\}$. Fourier expansions for functions $f \in C(C_2^2)$ are defined in terms of Walsh functions given by the columns of Walsh matrix

$$\mathbf{W}(2) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}. \quad (3)$$

The set of Walsh functions $w_i, i = 0, 1, 2, 3$ can be represented in terms of switching variables as

1. $w_0 = 1$,
2. $w_1 = 1 - 2x_2$,
3. $w_2 = 1 - 2x_1$,
4. $w_3 = (1 - 2x_1)(1 - 2x_2)$.

In symbolic notation, $\mathbf{W}(2)$ can be written as

$$\mathbf{X}(2) = [w_0 \quad w_1 \quad w_2 \quad w_3].$$

From (3), the Walsh expression for $f \in C(C_2^2)$ is defined by

$$\begin{aligned} f &= \mathbf{X}(2)\mathbf{S}_{w,f} \\ &= [1 \quad 1 - 2x_2 \quad 1 - 2x_1 \quad (1 - 2x_1)(1 - 2x_2)]\mathbf{S}_f, \end{aligned}$$

where $\mathbf{S}_{w,f} = [S_{w,f}(0), S_{w,f}(1), S_{w,f}(2), S_{w,f}(3)]^T$ is the vector of Walsh spectral coefficients.

With this notation, the orthogonal Walsh series expression transfers into the Walsh polynomial expressions in terms of switching variables. It is assumed that switching variables are coded by $(0, 1)_{GF(2)} \rightarrow (0, 1)_Z$. Therefore,

$$\begin{aligned} f &= 1 \cdot S_{w,f}(0) + (1 - 2x_2)S_{w,f}(1) + (1 - 2x_1)S_{w,f}(2) \\ &\quad + (1 - 2x_1)(1 - 2x_2)S_{w,f}(3). \end{aligned} \quad (4)$$

In (4), if the multiplications are performed, then the polynomial expression for f is derived.

$$f = 1 \cdot z_0 - 2x_1z_2 - 2x_2z_1 + 4x_1x_2z_3,$$

where

$$z_0 = S_{w,f}(0) + S_{w,f}(1) + S_{w,f}(2) + S_{w,f}(3),$$

$$z_1 = S_{w,f}(1) + S_{w,f}(3),$$

$$z_2 = S_{w,f}(2) + S_{w,f}(3),$$

$$z_3 = S_{w,f}(3).$$

Note that in this relation, indices of variables and coefficients are ordered in a way that corresponds to the Hadamard ordering of Walsh functions [9].

If Walsh (Fourier) coefficients are expressed in terms of the function values for f , these polynomial representations become the arithmetic expressions for f .

In this example,

$$S_{w,f}(0) = \frac{1}{4}(f(0) + f(1) + f(2) + f(3)),$$

$$S_{w,f}(1) = \frac{1}{4}(f(0) - f(1) + f(2) - f(3)),$$

$$S_{w,f}(2) = \frac{1}{4}(f(0) + f(1) - f(2) - f(3)),$$

$$S_{w,f}(3) = \frac{1}{4}(f(0) - f(1) - f(2) + f(3)).$$

Therefore,

$$z_0 = f(0),$$

$$z_1 = \frac{1}{2}(f(0) - f(1)),$$

$$z_2 = \frac{1}{2}(f(0) - f(2)),$$

$$z_3 = \frac{1}{4}(f(0) - f(1) - f(2) + f(3)).$$

After replacement of z_i in (4), we get the arithmetic expressions for $f \in C(C_2^n)$

$$f = 1 \cdot a_0 + x_1 a_2 + x_2 a_1 + x_1 x_2 a_3,$$

where

$$a_0 = f(0),$$

$$a_1 = f(0) - f(1),$$

$$a_2 = f(0) - f(2),$$

$$a_3 = f(0) - f(1) - f(2) + f(3).$$

If further

1. Binary values 0 and 1 for variables are considered as the logic values 0 and 1,
2. The addition and subtraction in C are replaced by the addition in $GF(2)$,
3. Values of coefficients are calculated modulo 2,

then, the arithmetic expressions become the Reed-Muller expressions for f .

In this example, the Reed-Muller expression is given by

$$f = 1 \cdot f_0 \oplus r_2 x_1 \oplus x_2 r_1 \oplus r_3 x_1 x_2,$$

where $r_i \in \{0, 1\}$, are

$$r_0 = f(0),$$

$$r_1 = f(0) \oplus f(1),$$

$$r_2 = f(0) \oplus f(2),$$

$$r_3 = f(0) \oplus f(1) \oplus f(2) \oplus f(3).$$

Example 5 For $n = 3$, the Walsh functions in the Hadamard ordering can be represented as

$$w_0(x) = 1,$$

$$w_1(x) = 1 - 2x_3,$$

$$w_2(x) = 1 - 2x_2,$$

$$w_3(x) = (1 - 2x_2)(1 - x_3),$$

$$w_4(x) = 1 - 2x_1,$$

$$w_5(x) = (1 - 2x_1)(1 - 2x_3),$$

$$w_6(x) = (1 - 2x_1)(1 - 2x_2),$$

$$w_7(x) = (1 - 2x_1)(1 - 2x_2)(1 - 2x_3).$$

If we replace these relations in (2), then

$$\begin{aligned} f &= \frac{1}{8}(4 - 2(1 - 2x_1) + 2(1 - 2x_1)(1 - 2x_3) \\ &\quad + 2(1 - 2x_1)(1 - 2x_2) + 2(1 - 2x_1)(1 - 2x_2)(1 - 2x_3)) \\ &= 1 - x_1 - x_2 - x_3 + 2x_1x_2 + 2x_1x_3 + x_2x_3 - 2x_1x_2x_3, \end{aligned}$$

which is the arithmetic expression for f .

If we reduce the coefficients modulo 2, and replace the addition and subtraction by EXOR, we get the Reed-Muller expression for f

$$f = 1 \oplus x_1 \oplus x_2 \oplus x_3 \oplus x_2x_3.$$

5 Arithmetic Expressions on Q_8

Let G be the Quaternion (non-Abelian) group Q_8 . The columns of the matrix Q_8 in terms of which the Fourier transform on Q_8 is defined can be represented by the following set of functions in terms of switching variables

1. $q_0 = 1$,
2. $q_1 = (1 - 2x_1)$,
3. $q_2 = (1 - 2x_3)$,
4. $q_3 = (1 - 2x_1)(1 - 2x_3)$,
5. $q_4 = \bar{x}_1(1 - 2x_2)(\bar{x}_3 + ix_3)$,
6. $q_5 = -x_1(1 - 2x_2)(\bar{x}_3 + ix_3)$,
7. $q_6 = x_1(1 - 2x_2)(\bar{x}_3 - ix_3)$,
8. $q_7 = \bar{x}_1(1 - 2x_2)(\bar{x}_3 - ix_3)$.

Each $f \in C(Q_8)$ can be represented as a linear combination of these functions. The coefficients in this expression are the Fourier coefficients on Q_8 over C .

Example 6 Consider a three-variable switching function f as a function on Q_8 . If f is given by the truth-vector $\mathbf{F} = [1, 0, 0, 0, 0, 1, 1, 1]^T$, then the Fourier coefficients for f are given by the vector $\mathbf{S}_f = \frac{1}{2}[4, -2, 0, 2, 2, 2, -2, 2]^T$. Therefore,

$$f = \frac{1}{8}(4 - 2(1 - 2x_1) + 2(1 - 2x_1)(1 - 2x_2) + 2\bar{x}_1(1 - 2x_2)(\bar{x}_3 + ix_3) - 2x_1(1 - 2x_2)(\bar{x}_3 + ix_3) - 2x_1(1 - 2x_2)(\bar{x}_3 - ix_3) + 2x_1(1 - 2x_2)(\bar{x}_3 - ix_3)).$$

Finally,

$$f = \frac{1}{8}(4 - 4x_3 + 8x_1x_3 + 4\bar{x}_1\bar{x}_3 - 4x_1\bar{x}_3 - 8\bar{x}_1x_2\bar{x}_3 + 8x_1x_2\bar{x}_3). \quad (5)$$

In a general case,

$$f = q_0S_{q,f}(0) + q_1S_{q,f}(1) + q_2S_{q,f}(2) + q_3S_{q,f}(3) + q_4S_{q,f}(4) + q_5S_{q,f}(5) + q_6S_{q,f}(6) + q_7S_{q,f}(7).$$

From the matrix of the group representations on Q_8 , the Fourier coefficients on Q_8 are given by

$$S_{q,f}(0) = \frac{1}{8}(f(0) + f(1) + f(2) + f(3) + f(4) + f(5) + f(6) + f(7))$$

$$S_{q,f}(1) = \frac{1}{8}(f(0) + f(1) + f(2) + f(3) - f(4) - f(5) - f(6) - f(7))$$

$$S_{q,f}(2) = \frac{1}{8}(f(0) - f(1) + f(2) - f(3) + f(4) - f(5) + f(6) - f(7))$$

$$S_{q,f}(3) = \frac{1}{8}(f(0) - f(1) + f(2) - f(3) - f(4) + f(5) - f(6) + f(7))$$

$$S_{q,f}(4) = \frac{1}{8}(2f(0) - 2if(1) - 2f(2) + 2if(3))$$

$$S_{q,f}(5) = \frac{1}{8}(-2f(4) + 2if(5) + 2f(6) - 2if(7))$$

$$S_{q,f}(6) = \frac{1}{8}(2f(4) + 2if(5) - 2f(6) - 2if(7))$$

$$S_{q,f}(7) = \frac{1}{8}(2f(0) + 2if(1) - 2f(2) - 2if(3))$$

From there,

$$\begin{aligned} f &= \frac{1}{2}(f(0) + f(2) + x_3(-f(0) + f(1) - f(2) + f(3)) \\ &\quad + x_1(-f(0) - f(2) + f(4) + f(6)) \\ &\quad + x_1x_3(f(0) - f(1) + f(2) - f(3) - f(4) + f(5) - f(6) + f(7)) \\ &\quad + \bar{x}_1\bar{x}_3(1 - 2x_2)(f(0) - f(2)) \\ &\quad + \bar{x}_1x_3(1 - 2x_2)(f(1) - f(3)) \\ &\quad + x_1\bar{x}_3(1 - 2x_2)(f(4) - f(6)) \\ &\quad + x_1x_3(1 - 2x_2)(f(5) - f(7)). \end{aligned}$$

In matrix notation, this expression can be considered as a series expansion in terms of the basic functions in $C(Q_8)$ represented by columns of the matrix

$$\mathbf{X}_q = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

If $f \in C(Q_8)$ is given by a vector $\mathbf{F} = [f(0), \dots, f(7)]^T$, then the coefficients $\mathbf{Q}_f = [q_0, \dots, q_7]^T$ in the expansion for f with respect to the basis given by \mathbf{X}_q are determined as

$$\mathbf{Q}_f = \mathbf{X}_q^{-1}\mathbf{F},$$

where \mathbf{X}_q^{-1} is the matrix inverse for \mathbf{X}_q over C . Thus,

$$\mathbf{X}_q^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{bmatrix}.$$

This matrix can be considered as a transform matrix defined with respect to the basic functions represented by columns of the matrix \mathbf{X}_q . Thus, this matrix defines the inverse transform. We call this transform the Arithmetic-Haar transform, since the matrix \mathbf{X}_q^{-1} exhibits a form similar to that of the Haar transform matrix [9].

Example 7 For f in Example 2, the arithmetic-Haar coefficients are given by the vector $\mathbf{S}_f = \frac{1}{2}[1, -1, 0, 2, 1, 0, -1, 0]^T$. Therefore,

$$f = \frac{1}{2}(1 - x_3 + 2x_1x_3 + \bar{x}_1\bar{x}_3(1 - 2x_2) - x_1\bar{x}_3(1 - 2x_2)).$$

Extensions of the arithmetic-Haar expressions from Q_8 to $Q_8^n, n = 3r$, can be done through the Kronecker product of the basic transform matrices on Q_8 in the same way as in the case of the arithmetic expressions and Walsh series expansions. Generalization to other values for n is easy, by using combinations of the basic arithmetic transform matrices. For example, if for a four variable switching function, the domain group is chosen as $G = C_2 \times Q_8$, then the arithmetic-Haar transform matrix is defined as the Kronecker product of $\mathbf{A}(1)$ and \mathbf{X}_q .

5.1 Arithmetic Expressions and Arithmetic-Haar Expressions

In derivation of arithmetic-Haar expressions, we have exploited the correspondence between the order of C_2^n and $Q_8^{n/3}$. Due to that, we express the basic functions in arithmetic-Haar expressions in terms of switching variables, similar as that is done for the arithmetic expression on C_2^n . Therefore, it is natural to establish a relationship between the arithmetic expressions on C_2^n and the arithmetic-Haar expressions on $Q_8^{n/3}$.

The arithmetic-Haar expression for a given function f , converts into the arithmetic expression for f if we replace $\bar{x}_i \rightarrow (1 - x_i)$, and then recalculate the coefficients and assign them to the basic functions used in the arithmetic expressions.

Example 8 For f in Example 7, the arithmetic-Haar expression is converted into the arithmetic expression as follows

$$\begin{aligned}
 f &= \frac{1}{2}(1 - x_3 + 2x_1x_3 + \bar{x}_1\bar{x}_3(1 - 2x_2) - x_1\bar{x}_3(1 - 2x_2)) \\
 &= \frac{1}{2}(1 - x_3 + 2x_1x_3 + (1 - x_1)(1 - x_3)(1 - 2x_2) - x_1(1 - x_3)(1 - 2x_2)) \\
 &= \frac{1}{2}(1 - x_3 + 2x_1x_3 + (1 - x_3)(1 - 2x_2) - 2x_1(1 - x_3)(1 - 2x_2)) \\
 &= 1 - x_3 - x_2 + x_2x_3 - x_1 + 2x_1x_2 + 2x_1x_3 - 2x_1x_2x_3.
 \end{aligned}$$

5.2 Arithmetic-Haar Expressions and Kronecker Expressions

If we use a different representations for functions q_0, \dots, q_7 , we will derive different arithmetic expressions for functions in $C(Q_8)$. We use the above given representations, since they

1. Resemble representations of Walsh functions in terms of switching variables,
2. Provide the transform matrix similar to the Haar transform.

Arithmetic expressions in $C(C_2^3)$ are series expansions in terms of basic functions generated as

$$[1 \ x_1] \otimes [1 \ x_2] \otimes [1 \ x_3].$$

If the values for variables are considered as the logic values, then the same basic functions are used in the definition of the Reed-Muller expressions. These basic functions can be generated by the recursive application of the positive Davio (pD) expansion defined as $f = 1 \cdot f_0 \oplus x_i(f_0 \oplus f_1)$, where $f_0 = f(x_i = 0)$, and $f_1 = f(x_i = 1)$.

Arithmetic-Haar expressions are series expansions in terms of the basic functions given by the symbolic matrix

$$\begin{aligned}
 \mathbf{X}_q &= [1 \ x_3 \ x_1 \ x_1x_3 \ \bar{x}_1\bar{x}_3(1 - 2x_2) \ \bar{x}_1x_3(1 - 2x_2) \\
 &\quad x_1\bar{x}_3(1 - 2x_2) \ x_1x_3(1 - 2x_2)].
 \end{aligned}$$

These basic functions can be considered as a combination of two sets of basic functions in $C(C_2^3)$. First four basic functions are generated by the pD-expansion for variables x_1 and x_3 . The other four basic functions are generated

by the Shannon (S) expansion defined by $f = \bar{x}_i f_0 \oplus x_i f_1$ for x_1 and x_3 , with multiplication by $(1 - 2x_2)$.

Thus, we generate the first four basic functions in the arithmetic-Haar expressions as

$$[1 \ x_1] \otimes [1 \ x_3] = [1 \ x_3 \ x_1 \ x_1 x_3].$$

For other four basic functions we first use the S-expansion and get

$$[\bar{x}_1 \ x_1] \otimes [\bar{x}_3 \ x_3] = [\bar{x}_1 \bar{x}_3 \ \bar{x}_1 x_3 \ x_1 \bar{x}_3 \ x_1 x_3].$$

Then, we multiply the generated terms by $(1 - 2x_2)$.

For functions in $GF_2(C_2^n)$, expressions generated by combination of pD-expansions and S-expansions are denoted as Kronecker expressions [19]. In this setting, the arithmetic-Haar expressions are modified integer counterparts of the Kronecker expressions for switching functions.

6 Different Polarity Polynomial Expressions

Switching variables take values in C_2 . Negative literals of switching variables are defined as $\bar{x} = x \oplus 1$. The fixed-polarity Reed-Muller expressions are defined by freely choosing between x and \bar{x} for each variable in f [20]. It is assumed that a particular variable appears as the positive or the negative literal, but not both in the same expressions for f . For a given f , fixed-polarity Reed-Muller expressions differ in the number of non-zero coefficients.

The polarity of variables chosen in an expression is conveniently expressed by the polarity vector $H = (h_1, \dots, h_n)$, $h_i \in \{0, 1\}$. It is assumed that if $h_i = 1$, then the i -th variable is represented by the negated literal \bar{x}_i . Thus, $h_i = 0$, shows that the i -th variable appears as x_i . The polarity where $h_i = 0$ for each $i = 1, \dots, n$ is denoted as the zero-polarity. For a given class of expressions and a given f , the polarity for which the expression has the minimal number of non-zero coefficients is denoted as the optimal polarity.

In Fourier expansions for functions in $C(Q_8)$, the basic functions can be expressed in terms of switching variables. Thanks to that, the same method can be used to define fixed-polarity Fourier expansions $C(Q_8)$, and the arithmetic-Haar expression for functions in $C(C_2^n)$ and $C(Q_8^{n/3})$, $C_2 Q_8^{(n-1)/3}$, and $C_2^2 Q_8^{(n-2)/3}$.

6.1 Fixed-polarity Fourier Expansions in $C(Q_2)$

Fixed-polarity Fourier expansions on Q_8 will be introduced through the example for $H = (101)$.

Example 9 The use of negated literals for variables x_1 and x_3 transfers the basic functions for the Fourier transform on Q_8 into

1. $q_0^{(101)} = 1,$
2. $q_1^{(101)} = 1 - 2\bar{x}_1,$
3. $q_2^{(101)} = 1 - 2\bar{x}_3,$
4. $q_3^{(101)} = (1 - 2\bar{x}_1)(1 - 2\bar{x}_3),$
5. $q_4^{(101)} = x_1(1 - 2x_2)(x_3 + i\bar{x}_3),$
6. $q_5^{(101)} = -\bar{x}_1(1 - 2x_2)(x_3 + i\bar{x}_3),$
7. $q_6^{(101)} = \bar{x}_1(1 - 2x_2)(x_3 - i\bar{x}_3),$
8. $q_7^{(101)} = x_1(1 - 2x_2)(x_3 - i\bar{x}_3).$

Thus, the matrix of basic functions \mathbf{Q}_8 for the Fourier transform in $C(Q_8)$ for the polarity $H = (101)$ is given by

$$\mathbf{Q}_8^{(101)} = \begin{bmatrix} 1 & -1 & -1 & 1 & 0 & i & -i & 0 \\ 1 & -1 & 1 & -1 & 0 & 1 & 1 & 0 \\ 1 & -1 & -1 & 1 & 0 & -i & i & 0 \\ 1 & -1 & 1 & -1 & 0 & -1 & -1 & 0 \\ 1 & 1 & -1 & -1 & i & 0 & 0 & -i \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & -1 & -1 & -i & 0 & 0 & i \\ 1 & 1 & 1 & 1 & -1 & 0 & 0 & -1 \end{bmatrix}.$$

It follows that the columns of $\mathbf{Q}_2^{(101)}$ are given by $\mathbf{q}_0, -\mathbf{q}_1, -\mathbf{q}_2, \mathbf{q}_3, i\mathbf{q}_6, i\mathbf{q}_7, -i\mathbf{q}_4, i\mathbf{q}_5$. This permutation and change of columns in $\mathbf{Q}_8^{(101)}$ requires a permutation and change of rows in \mathbf{Q}_8^{-1} . We first transform rows in \mathbf{Q}_8^{-1} in the same way as we transform the columns in \mathbf{Q}_8 to get $\mathbf{Q}_8^{(101)}$. In this way we get

$$(\mathbf{Q}_8^{-1})' = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 2i & -2 & -2i & 2 \\ 2i & -2 & -2i & 2 & 0 & 0 & 0 & 0 \\ -2i & -2 & 2i & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2i & -2 & 2i & 2 \end{bmatrix}.$$

Then, we permute the columns of $(\mathbf{Q}_8^{-1})'$ as is determined by the rule $x_1x_2x_3 \oplus \bar{h}_1\bar{h}_2\bar{h}_3$. Therefore, the columns of $(\mathbf{Q}_8^{-1})'$ are ordered as $q'_2, q'_3, q'_0, q'_1, q'_6, q'_7, q'_4, q'_5$. In this way, we get the matrix $(\mathbf{Q}_8^{(101)})^{-1}$ inverse to $\mathbf{Q}_8^{(101)}$. It is given by

$$(\mathbf{Q}_8^{(101)})^{-1} = \frac{1}{8} \begin{bmatrix} 1 & 1 & 1 & 1 & 11 & 1 & 1 \\ -1 & -1 & -1 & -1 & 11 & 1 & 1 \\ -1 & 1 & -1 & 1 & -11 & -1 & 1 \\ 1 & -1 & 1 & -1 & -11 & -1 & 1 \\ 0 & 0 & 0 & 0 & -2i & 2 & 2i & -2 \\ -2i & 2 & 2i & -2 & 00 & 0 & 0 & 0 \\ 2i & 2 & -2i & -2 & 00 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2i & 2 & -2i & -2 \end{bmatrix}.$$

For f in Example 2, the Fourier spectrum for the polarity $H = (101)$ is given by

$$\mathbf{S}_{q,f}^{(101)} = \frac{1}{8}[4, 2, 0, 2, 2i, -2i, 2i, -2i]^T.$$

Therefore,

$$\begin{aligned} f = \frac{1}{8} & (4 + 2(1 - 2\bar{x}_1) + 2(1 - 2\bar{x}_1)(1 - 2\bar{x}_3) + 2ix_1(1 - 2x_2)(x_3 + i\bar{x}_3) \\ & - 2i\bar{x}_1(1 - 2x_2)(x_3 + i\bar{x}_3) + 2i\bar{x}_1(1 - 2x_2)(x_3 - i\bar{x}_3) \\ & - 2ix_1(1 - 2x_2)(x_3 - i\bar{x}_3)). \end{aligned}$$

6.2 Fixed-polarity Arithmetic-Haar Expressions

The matrix of basic arithmetic-Haar functions does not have complex values, and therefore, the transformation of this matrix into the matrix for the given polarity H is simpler than in the case of the Fourier transform.

We will introduce the fixed-polarity arithmetic-Haar expressions in $C(Q_8)$ with the example for $H = (101)$.

Example 10 For $H = (101)$, the basic function for the arithmetic-Haar expressions in $C(Q_8)$ are given by columns of the matrix

$$\mathbf{X}_q^{(101)} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & -1 \\ 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}.$$

This matrix is obtained by the permutation of rows in \mathbf{X}_q by the rule $x_1x_2x_3 \oplus h_1h_2h_3$. Therefore, the rows with indices 0, 1, 2, 3, 4, 5, 6, 7 are permuted as 5, 4, 7, 6, 1, 0, 3, 2. This permutation requires the same permutation of columns in \mathbf{X}_q^{-1} . Therefore,

$$(\mathbf{X}_q^{(101)})^{-1} = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & 0 & -1 & 0 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

For f in Example 2, the arithmetic-Haar expression for the polarity $H = (101)$ is given by

$$f = \frac{1}{2}(2 - \bar{x}_3 - 2\bar{x}_1 + 2\bar{x}_1\bar{x}_3 - x_1\bar{x}_3(1 - 2x_2) + \bar{x}_1\bar{x}_3(1 - 2x_2)).$$

This expression requires six coefficients instead of five coefficients in the zero-polarity arithmetic-Haar expression for f .

Table 2 shows different spectra for f in Example 4, for the zero-polarity and the polarity $H = (101)$. Table 3 shows the corresponding expressions for f .

Table 4 compares the number of non-zero coefficients in some *mcnc* benchmark functions used in logic design for the Walsh (W), the arithmetic (A), the Fourier on Q_8 (F), and the arithmetic-Haar (AH) transform. For multiple-output functions, each output is considered as a separate switching function, and in this table, *fun-i* denotes the i -th output of *fun*. This table shows that for each transform we can find a function where this transform provides a spectrum with fewer number of non-zero coefficients. In the most cases, the arithmetic and the Walsh transform are the most efficient, since are applied to binary-valued functions. However, even the complex-valued Fourier transform although applied to binary-valued functions can produce simpler spectrum than these transforms. For example, for *sao2-i*, $i = 1, 2, 3, 4$, the Fourier transform is more convenient than both the arithmetic and the Walsh transform. However, in this case,

Table 2. Different spectra for f

Expression	Spectrum
RM	$[1, 1, 1, 1, 1, 0, 0, 0]^T$
$RM^{(101)}$	$[1, 1, 0, 1, 1, 0, 0, 1]^T$
A	$[1, -1, -1, 1, -1, 2, 2, -2]^T$
$A^{(101)}$	$[1, -1, 0, 1, -1, 2, 0, -2]^T$
AH	$\frac{1}{2}[1, -1, 0, 2, 1, 0, -1, 0]^T$
$AH^{(101)}$	$\frac{1}{2}[2, -1, -2, 2, 0, -1, 0, 1]^T$
W	$\frac{1}{8}[4, 0, 0, 0, -2, 2, 2, 2]^T$
$W^{(101)}$	$\frac{1}{8}[4, 0, 0, 0, 2, 2, -2, 2]^T$
F	$\frac{1}{8}[4, -2, 0, 2, 2, 2, -2, 2]^T$
$F^{(101)}$	$\frac{1}{8}[4, 2, 0, 2, 2i, -2i, 2i, -2i]^T$

Table 3. Different expressions for f

RM	$f = 1 \oplus x_3 \oplus x_2 \oplus x_2x_3 \oplus x_1$
$RM^{(101)}$	$f = 1 \oplus \bar{x}_3 \oplus x_2\bar{x}_3 \oplus \bar{x}_1 \oplus \bar{x}_1x_2\bar{x}_3$
A	$f = 1 - x_3 - x_2 + x_2x_3 - x_1 + 2x_1x_2 + 2x_1x_3 - 2x_1x_2x_3$
$A^{(101)}$	$f = 1 - \bar{x}_3 + x_2\bar{x}_3 - \bar{x}_1 + 2\bar{x}_1\bar{x}_3 - 2\bar{x}_1x_2\bar{x}_3$
AH	$f = \frac{1}{2}(1 - x_3 + 2x_1x_3 + \bar{x}_1\bar{x}_3(1 - 2x_2) - x_1\bar{x}_3(1 - 2x_2))$
$AH^{(101)}$	$f = \frac{1}{2}(2 - \bar{x}_3 - 2\bar{x}_1 + 2\bar{x}_1\bar{x}_3 - x_1\bar{x}_3(1 - 2x_2) + \bar{x}_1\bar{x}_3(1 - 2x_2))$
W	$f = \frac{1}{8}(4 - 2w_0(x) + 2w_5(x) + 2w_6(x) + 2w_7(x)), x = (x_1x_2x_3)$
$W^{(101)}$	$f = \frac{1}{8}(4 + 2(1 - 2\bar{x}_1) + 2(1 - 2\bar{x}_1)(1 - 2\bar{x}_3) - 2(1 - 2\bar{x}_1)(1 - 2x_2)(1 - 2\bar{x}_3))$
F	$f = \frac{1}{8}(4 - 2(1 - 2x_1) + 2(1 - 2x_1)(1 - 2x_2) + 2\bar{x}_1(1 - 2x_2)(\bar{x}_3 + ix_3) + 2x_1(1 - 2x_2)(\bar{x}_3 + ix_3) - 2x_1(1 - 2x_2)(\bar{x}_3 - ix_3) + 2\bar{x}_1(1 - 2x_2)(\bar{x}_3 - ix_3))$
$F^{(101)}$	$f = \frac{1}{8}(4 + 2(1 - 2\bar{x}_1) + 2(1 - 2\bar{x}_1)(1 - 2\bar{x}_3) + 2ix_1(1 - 2x_2)(x_3 + i\bar{x}_3) - 2i\bar{x}_1(1 - 2x_2)(x_3 + i\bar{x}_3) + 2i\bar{x}_1(1 - 2x_2)(x_3 - i\bar{x}_3) - 2ix_1(1 - 2x_2)(x_3 - i\bar{x}_3))$

Table 4. Number of non-zero coefficients

f	F	AH	W	A
5xp1-1	96	26	80	18
5xp1-2	122	50	128	33
5xp1-3	124	34	65	41
5xp1-4	70	56	33	27
5xp1-5	84	36	17	17
5xp1-6	64	44	9	9
5xp1-7	56	32	5	5
5xp1-8	28	20	2	3
5xp1-9	24	8	2	2
5xp1-10	64	10	128	7
9sym	464	300	256	465
rd84-1	220	189	256	191
rd84-2	160	52	2	255
rd84-3	48	4	256	1
rd84-4	237	166	256	163
sao2-1	202	200	1024	380
sao2-2	275	269	640	768
sao2-3	348	297	856	578
sao2-4	322	216	1024	936

the arithmetic-Haar transform further reduces the number of non-zero coefficients. Conversely, for a given function, we should try different transforms to determine a spectrum with minimum number of non-zero coefficients. For example, for rd84-2 and rd84-3, the Fourier transform produces spectra with fewer non-zero coefficients than the arithmetic and the Walsh transform, respectively. For rd84-3, the arithmetic-Haar spectrum is comparable to the most efficient transform. For 9sym, the arithmetic-Haar transform requires fewer non-zero coefficients than the arithmetic transform, and the Walsh transform is

the most efficient. For 5xp1-2, the arithmetic-Haar transform is more efficient than the Walsh transform, and the arithmetic transform is the most efficient. However, for 5xp1-3, the arithmetic-Haar transform is the most efficient. For all the outputs of sao2, the Fourier transform requires fewer products than either Walsh or the arithmetic transform, and the arithmetic-Haar transform is the most efficient.

We consider this analysis as a justification for introduction and use of different transforms, supported by a requirement that these transforms may share some useful properties of existing transforms. In that respect, the arithmetic-Haar transform is derived from the Fourier transform on Q_8^f in a uniform way as the arithmetic transform is derived from the Walsh transform on C_2^n . It eliminates the complex values in the Fourier transform and the corresponding transform matrix expresses a structure similar to the structure of the Haar matrix.

7 Calculation of the Arithmetic-Haar Coefficients

7.1 FFT-like Algorithm

The arithmetic-Haar transform matrix \mathbf{X}_q has a form similar to that of the Haar transform matrix. Thanks to that property, it is possible to derive a FFT-like algorithm for calculation of the arithmetic-Haar coefficients. This algorithm is based upon the following factorization of \mathbf{X}_q^{-1}

$$\mathbf{X}_q^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{bmatrix}.$$

Figure 1 compares the flow-graphs of the fast algorithms for calculation of the Walsh, the arithmetic, the Fourier, and the arithmetic-Haar coefficients for functions in $f \in C(C_2^3)$, and $f \in C(Q_8)$.

7.2 Calculation of Arithmetic-Haar Coefficients through Decision Diagrams

Decision diagrams (DDs) are an efficient data structure for compact representation of discrete functions [1], [4], [20]. Among many other applications, DDs are efficiently used to calculate different spectral transforms. From spectral interpretation of DDs [31], it was easy to show that in calculation of spectral transforms through DDs we actually perform basic operations in FFT-like algorithms. However, in DDs, calculation of the spectrum for f is not based on

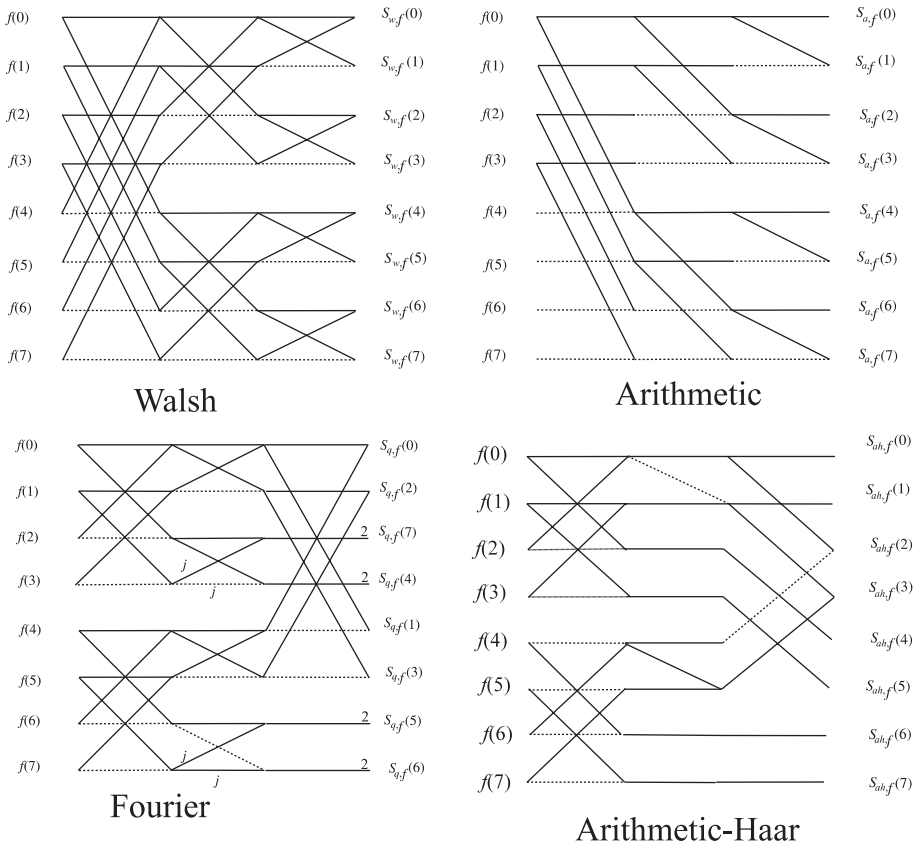


Fig. 1. Flow-graph for FFT-like algorithms for Walsh, arithmetic, Fourier and arithmetic-Haar coefficients for $n = 3$

the vector of function values, but is performed over the DD for f . That permits to use the particular properties of f , which provides the computation efficiency and possibility to process large functions [5], [20], [32].

Calculation of the coefficients in arithmetic expressions of switching functions through DDs was considered in several papers, see for example [6], [7], [17].

In this section, we present a method to calculate arithmetic-Haar coefficients through decision diagrams (DDs) [1], [4]. The method is derived as a suitable modification of the corresponding method for calculation of the Haar spectrum [22].

We assume that a given $f \in C(Q_8^r)$ is alternatively considered as a function in $C(C_2^n)$, $n = 3r$, and represented by a Binary DD [1], [4], or a Multi-terminal binary DD (MTBDD) [5], depending on the function values taken in $GF(2)$ or C , respectively.

The method will be first explained on decision trees (DTs), since DDs are derived by the reduction of DTs [20].

Example 11 Figure 2 shows BDT f in Example 2. Figure 3 shows the corresponding DD. In this DD, two cross points [31] are shown.

We will present the method in a general form for functions in $C(Q_8^n)$, but we illustrate the method by the example of functions $f \in C(Q_8)$.

We assign to each node two fields, where we write the result of calculation through DDs. Calculations consist of processing the nodes in the DT for f . In the case of DDs, cross points [31] should be also processed.

In a DD, the leftmost node at a level is the node which can be reached by the path labeled by a product of variables consisting of negated literals.

In a DT, values of constant nodes are function values for f . We assume that at the level $(n - 1)$ in the DT for f , the paths $\bar{x}_{n-1}x_n$ and $x_{n-1}\bar{x}_n$ are permuted. Thus, in the vector \mathbf{F} representing f , the pairs of adjacent elements, starting from $f(1)$ are permuted.

For $n = 3$, the values of f are given by $\mathbf{F} = [f(0), f(1), f(2), f(3), f(4), f(5), f(6), f(7)]^T$. After the mentioned reordering, the values of

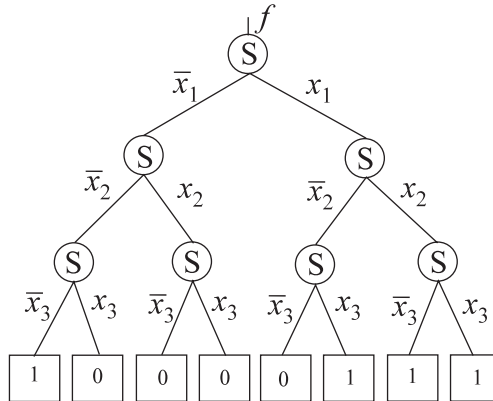


Fig. 2. BDT for f in Example 2

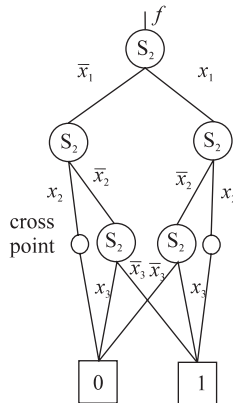


Fig. 3. BDD for F in Example 2

constant nodes in the DT for f are ordered as $\mathbf{F}_q = [f(0), f(2), f(1), f(3), f(4), f(6), f(5), f(7)]^T$.

The nodes at the level x_n are processed by using the rules described by rows of the matrix $\mathbf{W}(1)$. The values of the left and the right fields are determined by using the first row and the second row of $\mathbf{W}(1)$, respectively. The values of both fields for the nodes at all other levels are determined by using the rule described by the matrix $[-1, 1]$. The input data for calculation of values of left and right fields are determined as follows.

At the i -th level in the DT, if the node is pointed by the edge labeled with the negated literal \bar{x}_i , the input data for the left and right fields are reached by the subpaths $\bar{x}_i\bar{x}_{i+1}$ and \bar{x}_ix_{i+1} , respectively. If the node is pointed by the edge labeled with the positive literal, then the input data for the left and right fields are reached by the subpaths $\bar{x}_i\bar{x}_{i+1}$ and $x_i\bar{x}_{i+1}$, respectively.

The coefficient $S_{a,f}(0)$ is shown in the left field assigned to the leftmost node at the level x_n . The values of left fields assigned to the other nodes at this level are used in further calculations. The right fields of nodes at the level x_n show values of arithmetic-Haar coefficients $S_{a,f}(2^{n-1})$ to $S_{a,f}(2^n - 1)$. For $n = 3$, these are coefficients $S_{a,f}(4)$, $S_{a,f}(5)$, $S_{a,f}(6)$, and $S_{a,f}(7)$.

The other coefficients are shown in the left and right fields of the leftmost nodes from the level $n - 1$ to the root node. These coefficients are shown in the increasing order. Thus, the leftmost node at the level x_{n-1} shows $S_{a,f}(1)$, while the right field of this node shows $S_{a,f}(2)$. The root node shows the coefficient $S_{a,f}(2^{n-1} - 1)$. For $n = 3$, the leftmost nodes at level x_2 shows the values of $S_{a,f}(1)$ and $S_{a,f}(2)$. The root node shows $S_{a,f}(3)$.

Example 12 Figure 4 shows calculation of arithmetic-Haar coefficients for $f \in C(Q_8)$ though DT. Figure 5 shows calculation of arithmetic-Haar coefficients for f in Example 2 through BDT. Figure 6 shows the same calculation through BDD. Note that in this example, permutation of paths \bar{x}_2x_3 and $x_2\bar{x}_3$ does not permute values of constant nodes, since $f(1) = f(2) = 0$, $f(3) = f(4) = 0$, and $f(5) = f(6) = 1$.

This algorithm can be used to calculate arithmetic-Haar coefficients of complex-valued functions if instead of BDDs, we use MTBDDs.

8 Closing Remarks

In this paper, we first discussed derivation of the arithmetic expressions from the Walsh expansions. Then, we use the same method to derive the arithmetic expressions for functions in $C(Q_8)$. The resulting transform matrix has a structure similar to that of the Haar matrix. Thus, such representations are denoted as the arithmetic-Haar expressions. Different representations of basic functions in the Fourier expressions, produce different arithmetic expressions in $C(Q_8)$.

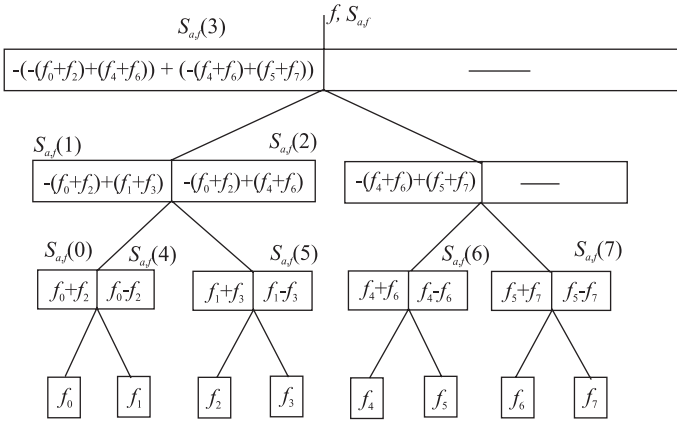


Fig. 4. Calculation of arithmetic-Haar coefficients through BDT

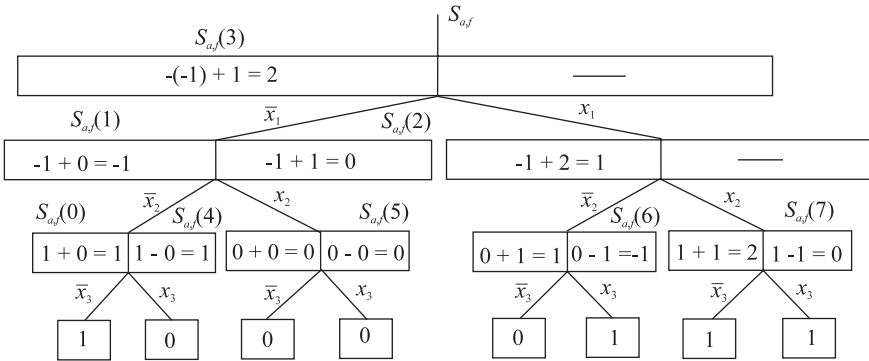


Fig. 5. Calculation of arithmetic-Haar coefficients for f in Example 2 through BDT

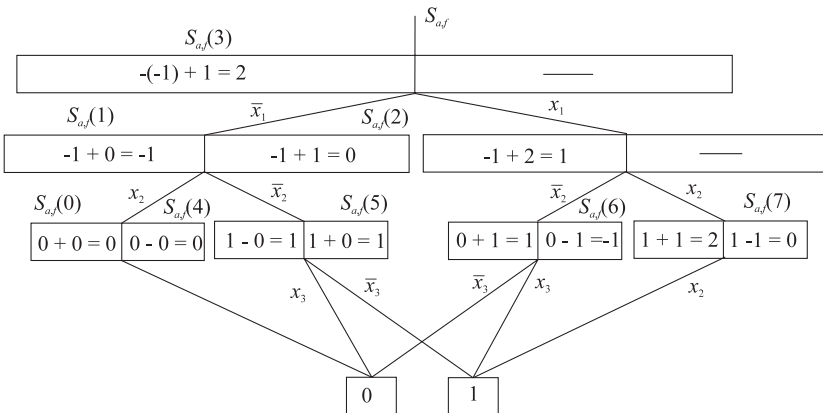


Fig. 6. Calculation of arithmetic-Haar coefficients for f in Example 2 through BDD

Extensions of these expressions to functions on groups Q_8 and arbitrary not necessarily Abelian groups can be easily done by using the same method.

Since the basic functions in the Fourier expansions in $C(Q_8)$ can be expressed in terms of the switching variables, we define the fixed-polarity Fourier expressions in $C(Q_8)$. In the same way, we define the fixed-polarity arithmetic-Haar expressions.

For the given $f \in C(C_2^n)$, the optimization of arithmetic expressions is performed by choosing different polarity for variables. Another way of optimization is to choose among different basic functions to derive polynomial expressions. For example, we can choose among the Reed-Muller, arithmetic, Walsh expressions, etc.

The optimization of arithmetic expressions for a given f is performed by choosing different polarity for variables. The considerations in this paper, offer another way for optimization of expressions for f . We can choose different domain groups for f , C_2^n or $Q_8^{n/3}$, and then perform optimization of the representations as in the case of functions on $C(C_2^n)$. Thus, by choosing among different sets of basic functions, and by different polarity of variables in each of these expressions. Therefore, we can perform optimization of the arithmetic expressions for a given f by choosing different

1. Polarity for variables,
2. Domain groups,
3. Representation of basic functions in the Fourier transform from which the arithmetic expressions are derived.

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