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A Concise Guide to Complex Hadamard Matrices

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Abstract. Complex Hadamard matrices, consisting of unimodular entries with arbitrary phases, play an important role in the theory of quantum information. We review basic properties of complex Hadamard matrices and present a catalogue of inequivalent cases known for the dimensions $N = 2, \ldots, 16$. In particular, we explicitly write down some families of complex Hadamard matrices for $N = 12, 14$ and 16, which we could not find in the existing literature.

1. Introduction

In a 1867 paper on 'simultaneous sign-successions, tessellated pavements in two or more colors, and ornamental tile-work' Sylvester used *self-reciprocial* matrices, defined as a square array of elements of which each is proportional to its first minor [1]. This wide class of matrices includes in particular these with orthogonal rows and columns. In 1893 Hadamard proved that such matrices attain the largest value of the determinant among all matrices with entries bounded by unity [2]. After that paper the matrices with entries equal to ± 1 and mutually orthogonal rows and columns were called *Hadamard*.

Originally there was an interest in *real* Hadamard matrices, $H_{ij} \in \mathbb{R}$, which found diverse mathematical applications, in particular in error correction and coding theory [3] and in the design of statistical experiments [4]. Hadamard proved that such matrices may exist only for $N = 2$ or for size N being a multiple of 4 and conjectured that they exist for all such N. A huge collection of real Hadamard matrices for small values of N is known (see e.g. [5, 6]), but the original Hadamard conjecture remains unproven. After a recent discovery of $N = 428$ real Hadamard matrix [7], the case $N = 668$ is the smallest order, for which the existence problem remains open.

Real Hadamard matrices may be generalized in various ways. Butson introduced a set $H(q,N)$ of Hadamard matrices of order N, the entries of which are q-th roots of unity [8, 9]. Thus $H(2,N)$ represents real Hadamard matrices, while $H(4,N)$ denotes Hadamard matrices[1] with entries ± 1 or $\pm\, i$. If p is prime, then $H(p,N)$ can exist if $N = mp$ with an integer m [8], and it is conjectured that they exist for all such cases [14].

In the simplest case $m = 1$ Hadamard matrices $H(N,N)$ exist for any dimension. An explicit construction based on *Fourier matrices*,

$$\left[F'_N\right]_{j,k} := \frac{1}{\sqrt{N}} e^{i(j-1)(k-1)\frac{2\pi}{N}} \quad \text{with} \quad j,k \in \{1,\ldots,N\}\,, \tag{1}$$

works for an arbitrary N, not necessarily prime. A Fourier matrix is unitary, but a rescaled matrix $F_N = \sqrt{N} F'_N$ belongs to $H(N,N)$. In the following we are going to use Hadamard matrices with unimodular entries, but for convenience we shall also refer to F_N as Fourier matrix.

For $m = 2$ it is not difficult [8] to construct matrices $H(p,2p)$. In general, the problem of finding all pairs $\{q,N\}$ for which Butson-type matrices $H(q,N)$ do exist, remains unsolved [15], even though some results on non-existence are available [16]. The set of p-th roots of unity forms a finite group and it is possible to generalize the notion of Butson-type matrices for any finite group [17, 18, 19].

In this work we will be interested in a more general case of *complex Hadamard matrices*, for which there are no restrictions on phases of each entry of H. Such a matrix, also called *biunitary* [20], corresponds to taking the limit $q \to \infty$ in the definition of Butson. In this case there is no analogue of Hadamard conjecture, since complex Hadamard matrices exists for any dimension. However, in spite of many years of research, the problem of finding all complex Hadamard matrices of a given size N is still open [21, 22].

Note that this problem may be considered as a particular case of a more general issue of specifying all unitary matrices such that their squared moduli give a fixed doubly stochastic matrix [23, 24]. This very problem was intensively studied by high energy physicists investigating the parity violation and analyzing the Cabibbo-Kobayashi-Maskawa matrices [25 − 28].

On one hand, the search for complex Hadamard matrices is closely related to various mathematical problems, including construction of some $*$-subalgebras in finite von Neumann algebras [21, 29 − 31], analyzing bi-unimodular sequences or finding cyclic n-roots [32, 33] and equiangular lines [34]. Complex Hadamard matrices were used to construct error correcting codes [15] and to investigate the spectral sets and Fuglede's conjecture [35 − 38].

On the other hand, Hadamard matrices find numerous applications in several problems of theoretical physics. For instance, Hadamard matrices (rescaled by $1/\sqrt{N}$ to achieve unitarity), are known in quantum optics as symmetric multiports [40, 41] (and are sometimes called Zeilinger matrices) and may be used to construct spin models [34], or schemes of selective coupling of a multi-qubit system [42].

Complex Hadamard matrices play a crucial role in the *theory of quantum information* as shown in a seminal paper of Werner [43]. They are used in solving

[1] These matrices were also called *complex* Hadamard matrices [10 − 13].

the Mean King Problem [44, 45, 46], and in finding 'quantum designs' [47]. Furthermore, they allow one to construct

a) *Bases of unitary operators*, i.e. the set of mutually orthogonal unitary operators, $\{U_k\}_{k=1}^{N^2}$ such that $U_k \in \mathcal{U}(N)$ and $\mathrm{Tr}\, U_k^\dagger U_l = N\delta_{kl}$ for $k, l = 1, \ldots, N^2$,
b) *Bases of maximally entangled states*, i.e. the set $\{|\Psi_k\rangle\}_{k=1}^{N^2}$ such that each $|\Psi_k\rangle$ belongs to a composed Hilbert space with the partial trace

$$\mathrm{Tr}_N(|\Psi_k\rangle\langle\Psi_k|) \;=\; \frac{\mathbb{1}}{N}\,,$$

and they are mutually orthogonal, $\langle\Psi_k|\Psi_l\rangle = \delta_{kl}$ [48],
c) *Unitary depolarisers*, i.e. the set $\{U_k\}_{k=1}^{N^2}$ such that for any bounded linear operator A the property $\sum_{k=1}^{N^2} U_k^\dagger A U_k = N\, \mathrm{Tr}\, A\mathbb{1}$ holds.

The problems a) − c) are equivalent in the sense that given a solution to one problem one can find a solution to the other one, as well as a corresponding scheme of teleportation or dense coding [43]. In particular Hadamard matrices are useful to construct a special class of unitary bases of a group type, also called 'nice error basis' [49, 50]

Another application of Hadamard matrices is related to quantum tomography: To determine all $N^2 - 1$ parameters characterizing a density matrix of size N one needs to perform $k \geq N + 1$ orthogonal measurements. Each measurement can be specified by an orthogonal basis $\Phi_u = \{|\phi_i^{(u)}\rangle\}_{i=1}^N$ set for $u = 1, \ldots, k$. Precision of such a measurement scheme is optimal if the bases are *mutually unbiased*, i.e. they are such that

$$\left|\langle\phi_i^{(u)}|\phi_j^{(s)}\rangle\right|^2 \;=\; \frac{1}{N}(1 - \delta_{us}) + \delta_{us}\delta_{ij}\,. \tag{2}$$

If the dimension N is prime or a power of prime the number of maximally unbiased bases (MUBs) is equal to $N + 1$ [51, 52], but for other dimensions the answer to this question is still unknown [53, 54, 55]. The task of finding $(k + 1)$ MUBs is equivalent to finding a collection of k *mutually unbiased Hadamards* (MUH),

$$\left\{H_i \in \mathcal{H}_N\right\}_{i=1}^k \;:\quad \frac{1}{\sqrt{N}} H_i^\dagger H_j \in \mathcal{H}_N, \quad i > j = 1, \ldots, k-1\,, \tag{3}$$

since the set $\{\mathbb{1}, H_1/\sqrt{N}, \ldots, H_k/\sqrt{N}\}$ forms a set of MUBs. Here $\mathcal{H}_N$ denotes the set of complex[2] Hadamard matrices of size N.

The aim of this work is to review properties of complex Hadamard matrices and to provide a handy collection of these matrices of size ranging from 2 to 16. Not only do we list concrete Hadamard matrices, the existence of which follows from recent papers by Haagerup [21] and Diţă [22], but also we present several other Hadamard matrices which have not appeared in the literature so far.

[2]Similarly, knowing unbiased real Hadamard matrices one constructs real MUB's [56].

2. Equivalent Hadamard Hatrices and the Dephased Form

We shall start this section providing some formal definitions.

DEFINITION 2.1 A square matrix H of size N consisting of unimodular entries, $|H_{ij}| = 1$, is called a *Hadamard matrix* if

$$HH^{\dagger} = N\,\mathbb{1}\,, \tag{4}$$

where † denotes the Hermitian transpose. One distinguishes

a) *real Hadamard matrices*, $H_{ij} \in \mathbb{R}$, for $i, j = 1, \ldots, N$,

b) *Hadamard matrices of Butson type* $H(q, N)$, for which $(H_{ij})^q = 1$,

c) *complex Hadamard matrices*, $H_{ij} \in \mathbb{C}$.

The set of all complex Hadamard matrices of size N will be denoted by $\mathcal{H}_N$.

DEFINITION 2.2 Two Hadamard matrices H_1 and H_2 are called *equivalent*, written $H_1 \simeq H_2$, if there exist diagonal unitary matrices D_1 and D_2 and permutations matrices P_1 and P_2 such that [21]

$$H_1 \;=\; D_1 P_1\, H_2\, P_2 D_2\,. \tag{5}$$

This equivalence relation may be considered as a generalization of the Hadamard equivalence in the set of real Hadamard matrices, in which permutations and negations of rows and columns are allowed.[3]

DEFINITION 2.3 A complex Hadamard matrix is called *dephased* when the entries of its first row and column are all equal to unity[4],

$$H_{1,i} \;=\; H_{i,1} \;=\; 1 \quad \text{for} \quad i = 1, \ldots, N\,. \tag{6}$$

Remark 2.1 For any complex $N \times N$ Hadamard matrix H there exist uniquely determined diagonal unitary matrices, $D_r = \mathrm{diag}(\bar{H}_{11}, \bar{H}_{21}, \ldots, \bar{H}_{N1})$, and $D_c = \mathrm{diag}(1, H_{11}\bar{H}_{12}, \ldots, H_{11}\bar{H}_{1N})$, such that $[D_c]_{1,1} = 1$ and

$$D_r \cdot H \cdot D_c \tag{7}$$

is dephased.

Two Hadamard matrices with the same dephased form are equivalent. Thus the relevant information on a Hadamard matrix is carried by the lower right submatrix of size $N - 1$, called the *core* [9].

[3] Such an equivalence relation may be extended to include also transposition and complex conjugation [28]. Since the transposition of a matrix is not realizable in physical systems we prefer to stick to the original definition of equivalence.

[4] In case of real Hadamard matrices such a form is called *normalised*.

It is often useful to define a *log-Hadamard* matrix Φ, such that

$$H_{kl} \ = \ e^{i\Phi_{kl}} \,, \tag{8}$$

is Hadamard. The *phases* Φ_{kl} entering a log-Hadamard matrix may be chosen to belong to $[0, 2\pi)$. This choice of phases implies that the matrix $q\Phi/2\pi$ corresponding to a Hadamard matrix of the Butson type $H(q, N)$ consists of zeros and integers smaller than q. All the entries of the first row and column of a log Hadamard matrix Φ, corresponding to a dephased Hadamard matrix, are equal to zero.

To illustrate the procedure of dephasing consider the Fourier-like matrix of size four, $[\widetilde{F}_4]_{jk} := e^{ijk2\pi/4}$ where $j, k \in \{1, 2, 3, 4\}$. Due to this choice of entries the matrix $\widetilde{F}_4$ is not dephased, but after operation (7) it takes the dephased form F_4,

$$\widetilde{F}_4 = \begin{bmatrix} i & -1 & -i & 1 \\ -1 & 1 & -1 & 1 \\ -i & -1 & i & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \qquad F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}. \tag{9}$$

The corresponding log-Hadamard matrices read

$$\widetilde{\Phi}_4 = \frac{2\pi}{4} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 0 & 2 & 0 \\ 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \Phi_4 = \frac{2\pi}{4} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 0 & 2 \\ 0 & 3 & 2 & 1 \end{bmatrix}. \tag{10}$$

Note that in this case dephasing is equivalent to certain permutation of rows and columns, but in general the role of both operations is different. It is straightforward to perform (7) which brings any Hadamard matrix into the dephased form. However, having two matrices in such a form it might not be easy to verify, whether there exist permutations P_1 and P_2 necessary to establish the equivalence relation (5). This problem becomes difficult for larger matrix sizes, since the number of possible permutations grows as $N!$. What is more, combined multiplication by unitary diagonal and permutation matrices may still be necessary.

To find necessary conditions for equivalence one may introduce various invariants of operations allowed in (5), and compare them for two matrices investigated. In the case of real Hadamard matrices and permutations only, the number of negative elements in the dephased form may serve for this purpose. Some more advanced methods for detecting inequivalence were recently proposed by Fang and Ge [57]. It has been known for many years that for $N = 4, 8$ and 12 all real Hadamard matrices are equivalent, while the number of equivalence classes for $N = 16, 20, 24$ and 28 is equal to $5, 3, 60$ and 487, respectively. For higher dimensions this number grows dramatically: For $N = 32$ and 36 the number of inequivalent matrices is not smaller than $3,578,006$ and $4,745,357$, but the problem of enumerating all of them remains open — see e.g. [58].

To characterize a complex Hadamard matrix H let us define a set of coefficients,

$$\Lambda \ = \ \Big\{ \Lambda_\mu := H_{ij}\overline{H}_{kj}H_{kl}\overline{H}_{il} \ : \ \mu = (i, j, k, l) \in \{1, \ldots, N\}^{\times 4} \Big\} , \tag{11}$$

where no summation over repeating indices is assumed and μ stands for a composite index (i, j, k, l). Due to complex conjugation of the even factors in the above definition any concrete value of Λ_μ is invariant with respect to multiplication by diagonal unitary matrices.[5] Although this value may be altered by a permutation, the entire set Λ is invariant with respect to operations allowed in (5). This fact allows us to state the Haagerup condition for nonequivalence [21],

LEMMA 2.1 *If two Hadamard matrices have different sets Λ of invariants (11), they are not equivalent.*

The above criterion works in one direction only. For example, any Hadamard matrix H and its transpose, H^T, possess the same sets Λ, but they need not to be permutation equivalent.

Some other equivalence criteria are dedicated to certain special cases. The tensor product of two Hadamard matrices is also a Hadamard matrix,

$$H_1 \in \mathcal{H}_M \quad \text{and} \quad H_2 \in \mathcal{H}_N \quad \Longrightarrow \quad H_1 \otimes H_2 \in \mathcal{H}_{MN} \tag{12}$$

and this fact will be used to construct Hadamard matrices of a larger size. Of particular importance are tensor products of Fourier matrices, $F_M \otimes F_N \in \mathcal{H}_{MN}$. For arbitrary dimensions both tensor products are equivalent, $F_N \otimes F_M \simeq F_M \otimes F_N$. However, their equivalence with F_{MN} depends on the number theoretic property of the product $M \cdot N$: the equivalence holds if M and N are relatively prime [59].

To classify and compare various Hadamard matrices we are going to use the dephased form (6) thus fixing the first row and column in each compared matrix. However, the freedom of permutation inside the remaining submatrix of size $(N-1)$ does not allow us to specify a unique 'canonical form' for a given complex Hadamard matrix. In other words we are going to list only certain representatives of each equivalence class known, but the reader has to be warned that other choices of equivalent representatives are equally legitimate. For instance, the one parameter family of $N = 4$ complex Hadamard matrices presented in [21, 22] contains all parameter-dependent elements of H in its lower-right corner, while our choice (65) with variable phases in second and fourth row is due to the fact that such an orbit stems from the Fourier matrix F_4.

Identifying matrices equivalent with respect to (5) we denote by

$$\mathcal{G}_N \;=\; \mathcal{H}_N/_{\simeq} \tag{13}$$

every set of representatives of different equivalence classes.

Interestingly $\mathcal{G}_N$ is known only for $N = 2, 3, 4, 5$, while the problem of finding *all* complex Hadamard matrices for $N \geq 6$ remains unsolved. In particular, compiling our list of Hadamard matrices we took into account all continuous families known to us, but in several cases it is not clear, whether there exist any equivalence relations between them.

[5] Such invariants were used by physicists investigating unitary Kobayashi-Maskawa matrices of size 3 and 4 [25 – 28].

3. Isolated Hadamard Matrices and Continuous Orbits of Inequivalent Matrices

In this and the following sections we shall use the symbol $\circ$ to denote the Hadamard product of two matrices,

$$[H_1 \circ H_2]_{i,j} \;=\; [H_1]_{i,j} \cdot [H_2]_{i,j}\,, \tag{14}$$

and the EXP symbol to denote the entrywise exp operation on a matrix,

$$[\mathrm{EXP}(R)]_{i,j} \;=\; \exp([R]_{i,j})\,. \tag{15}$$

3.1. Isolated Hadamard matrices

DEFINITION 3.1 A dephased $N \times N$ complex Hadamard matrix H is called *isolated* if there is a neighbourhood $\mathcal{W}$ around H such that there are no other dephased complex Hadamard matrices in $\mathcal{W}$.

To have a tool useful in determining whether a given dephased complex Hadamard matrix of size N is isolated, we introduce the notion of *defect*:

DEFINITION 3.2 The *defect* $d(H)$ of an $N \times N$ complex Hadamard matrix H is the dimension of the solution space of the real linear system with respect to a matrix variable $R \in \mathbb{R}^{N^2}$:

$$\left\{ \begin{array}{rcll} R_{1,j} & = & 0\,, & j \in \{2, \ldots, N\}\,, \quad (16) \\ R_{i,1} & = & 0\,, & i \in \{1, \ldots, N\}\,, \quad (17) \\ \sum_{k=1}^{N} H_{i,k}\overline{H}_{j,k}\,(R_{i,k} - R_{j,k}) & = & 0\,, & 1 \le i < j \le N\,. \quad (18) \end{array} \right.$$

The defect allows us to formulate a *one way* criterion:

LEMMA 3.1 *A dephased complex Hadamard matrix H is isolated if the defect of H is equal to zero.*

If N is prime then the defect of F_N is zero, so the Fourier matrix is isolated, as earlier shown in [20, 62]. For any composed N the defect of the Fourier matrix is positive. For instance, if the dimension N is a product of two distinct primes, then $d(F_{pq}) = 2(p-1)(q-1)$, while for powers of a prime, $N = p^k$ with $k \ge 2$, the defect reads

$$d(F_{p^k}) \;=\; p^{k-1}[k(p-1) - p] + 1\,. \tag{19}$$

An explicit formula for $d(F_N)$ for an arbitrary composed N is derived elsewhere [63]. In that case the Fourier matrix belongs to a continuous family, as examples show in Sect. 5.

The reasoning behind the criterion of Lemma 3.1 runs as follows:

Any dephased complex Hadamard matrices, in particular those in a neighbourhood of a dephased complex Hadamard matrix H, must be of the form

$$H \circ \mathrm{EXP}(\boldsymbol{i} \cdot R)\,, \tag{20}$$

where an $n \times n$ real matrix R satisfies 'dephased property' and unitarity conditions for (20):

$$\begin{cases} R_{1,j} = 0\,, & j \in \{2, \dots, N\}\,, \quad (21) \\ R_{i,1} = 0\,, & i \in \{1, \dots, N\}\,, \quad (22) \\ -\boldsymbol{i} \cdot \sum_{k=1}^{N} H_{i,k} \overline{H}_{j,k} e^{\boldsymbol{i}\left(R_{i,k} - R_{j,k}\right)} = 0\,, & 1 \le i < j \le N\,. \quad (23) \end{cases}$$

We will rewrite these conditions using a real vector function f, whose coordinate functions will be indexed by the values (symbol sequences) from the set $\mathcal{I}$, related to the standard index set by a fixed bijection (one to one map) $\beta :\ \mathcal{I} \longrightarrow \{1, 2, \dots, (2N-1) + (N^2 - N)\}$. The set $\mathcal{I}$ reads

$$\begin{aligned} \mathcal{I} \;=\; & \Big\{(1,2),\ (1,3),\ \dots\ ,\ (1,N)\Big\} \ \cup\ \Big\{(1,1),\ (2,1),\ \dots\ ,\ (N,1)\Big\} \\ & \cup \Big\{(i,j,t)\ :\ 1 \le i < j \le N \quad \text{and} \quad t \in \{\mathrm{Re},\ \mathrm{Im}\}\Big\}\,. \end{aligned} \tag{24}$$

For simplicity of notation we further write f_i, $i \in \mathcal{I}$ to denote $f_{\beta(i)}$.

Similarly, a fixed bijection $\alpha : \{1, \dots, N\} \times \{1, \dots, N\} \longrightarrow \{1, \dots, N^2\}$ allows matrix indexing the components of a real N^2 element vector variable R, an argument to f, and we write $R_{k,l}$ to denote $R_{\alpha(k,l)}$.

The $(2N-1)+(N^2-N)$ element function vector f is defined by the formulas:

$$\begin{cases} f_{(1,j)} = R_{1,j}\,, & j \in \{2, \dots, N\}\,, \quad (25) \\ f_{(i,1)} = R_{i,1}\,, & i \in \{1, \dots, N\}\,, \quad (26) \\ f_{(i,j,\mathrm{Re})} = \mathrm{Re}\Big(-\boldsymbol{i} \cdot \sum_{k=1}^{N} H_{i,k} \overline{H}_{j,k} e^{\boldsymbol{i}\left(R_{i,k} - R_{j,k}\right)}\Big)\,, & 1 \le i < j \le N\,. \quad (27) \\ f_{(i,j,\mathrm{Im})} = \mathrm{Im}\Big(-\boldsymbol{i} \cdot \sum_{k=1}^{N} H_{i,k} \overline{H}_{j,k} e^{\boldsymbol{i}\left(R_{i,k} - R_{j,k}\right)}\Big)\,, & 1 \le i < j \le N\,. \quad (28) \end{cases}$$

Conditions (21), (22), (23) can now be rewritten as

$$f(R) = \mathbf{0}\,, \tag{29}$$

where the $\alpha(i,j)$-th coordinate of a real variable vector R represents the i,j-th entry of the corresponding matrix R sitting in (20).

The value of the linear map $Df_{\mathbf{0}} : \mathbb{R}^{N^2} \longrightarrow \mathbb{R}^{(2N-1)+(N^2-1)}$, being the differential of f at $\mathbf{0}$, at R is the vector

$$[Df_{\mathbf{0}}(R)]_{(1,j)} \;=\; R_{1,j}\,, \qquad j \in \{2,\dots,N\}\,, \tag{30}$$

$$[Df_{\mathbf{0}}(R)]_{(i,1)} \;=\; R_{i,1}\,, \qquad i \in \{1,\dots,N\}\,, \tag{31}$$

$$[Df_{\mathbf{0}}(R)]_{(i,j,\mathrm{Re})} \;=\; \mathrm{Re}\Big(\sum_{k=1}^{N} H_{i,k}\overline{H}_{j,k}\,(R_{i,k}-R_{j,k})\Big)\,, \qquad 1 \le i < j \le N\,, \tag{32}$$

$$[Df_{\mathbf{0}}(R)]_{(i,j,\mathrm{Im})} \;=\; \mathrm{Im}\Big(\sum_{k=1}^{N} H_{i,k}\overline{H}_{j,k}\,(R_{i,k}-R_{j,k})\Big)\,, \qquad 1 \le i < j \le N\,, \tag{33}$$

where again indexing for f defined by β is used.

It is clear now that the kernel of the differential, $\{R \in \mathbb{R}^{N^2} \;:\; Df_{\mathbf{0}}(R) = \mathbf{0}\}$, corresponds to the solution space of system (16), (17), (18), in which R now takes the meaning of an input variable vector to f, with indexing determined by α.

Note that the (N^2-N)-equation subsystem (18)

$$\sum_{k=1}^{N} H_{i,k}\overline{H}_{j,k}\,(R_{i,k}-R_{j,k}) \;=\; 0\,, \qquad 1 \le i < j \le N\,, \tag{34}$$

is solved at least by the $(2N-1)$-dimensional real space spanned by $2N-1$ vectors, defined by

$$R_{k,l} \;=\; \begin{cases} 1 & \text{for } (k,l) \in \{1,\dots,N\} \times \{j\} \\ 0 & \text{otherwise,} \end{cases} \qquad j \in \{2,\dots,N\}\,, \tag{35}$$

and

$$R_{k,l} \;=\; \begin{cases} 1 & \text{for } (k,l) \in \{i\} \times \{1,\dots,N\} \\ 0 & \text{otherwise,} \end{cases} \qquad i \in \{1,\dots,N\}\,, \tag{36}$$

that is, by vectors, if treated as matrices (with the i,j-th entry being equal to the $\alpha(i,j)$-th coordinate of the corresponding variable vector), forming matrices with either a row or a column filled all with 1's and the other entries being 0's.

If the defect of H equals 0, then the overall system (16), (17), (18) is solved only by $\mathbf{0}$, the differential $Df_{\mathbf{0}}$ has full rank N^2, i.e. $\dim(Df_{\mathbf{0}}(\mathbb{R}^{N^2})) = N^2$, and we can choose an N^2 equation subsystem

$$\widetilde{f}(R) \;=\; \mathbf{0} \tag{37}$$

of (29) such that the differential $D\widetilde{f}_{\mathbf{0}}$ at $\mathbf{0}$ is of rank N^2, i.e. $\dim(D\widetilde{f}_{\mathbf{0}}(\mathbb{R}^{N^2})) = N^2$, and thus $\widetilde{f}$ satisfies the inverse function theorem. The theorem implies in our case that, in a neighbourhood of $\mathbf{0}$, $R = \mathbf{0}$ is the only solution to (37), as well as the only solution to (29).

Also in this case, let us consider the differential, at $\mathbf{0}$, of the partial function vector $f^{\mathcal{U}}$, given by (27), (28). This differential value at R, $Df^{\mathcal{U}}{}_{\mathbf{0}}(R)$, is given by the partial vector (32), (33). Since the 'remaining' differential, corresponding to the dephased 'property condition' part of f, defined by (30) and (31), is of rank $(2N-1)$, the rank of $Df^{\mathcal{U}}{}_{\mathbf{0}}$ is equal to $N^2-(2N-1)$. Recall that from considering above the minimal solution space of system (18), it cannot be greater than $N^2-(2N-1)$. Were it smaller, the rank of $Df_{\mathbf{0}}$ would be smaller than N^2, which cannot be if the defect of H is 0, see above.

Then one can choose an $N^2-(2N-1)$ equation subsystem $\widetilde{f}^{\mathcal{U}}(R)=\mathbf{0}$ of system $f^{\mathcal{U}}(R)=\mathbf{0}$, with the full rank

$$\dim(D\widetilde{f}^{\mathcal{U}}{}_{\mathbf{0}}(\mathbb{R}^{N^2})) \ = \ N^2-(2N-1)\,, \tag{38}$$

thus defining a $(2N-1)$ dimensional manifold around $\mathbf{0}$. This manifold generates, by (20), the $(2N-1)$ dimensional manifold containing all, not necessarily dephased, complex Hadamard matrices in a neighbourhood of H. In fact, the latter manifold is equal, around H, to the $(2N-1)$-dimensional manifold of matrices obtained by left and right multiplication of H by unitary diagonal matrices,

$$\Big\{\mathrm{diag}\,(e^{\boldsymbol{i}\alpha_1},\ldots,e^{\boldsymbol{i}\alpha_N})\cdot H\cdot \mathrm{diag}\,(1,e^{\boldsymbol{i}\beta_2},\ldots,e^{\boldsymbol{i}\beta_N})\Big\} \tag{39}$$

3.2. Continuous orbits of Hadamard matrices

The set of inequivalent Hadamard matrices is finite for $N=2$ and $N=3$, but already for $N=4$ there exists a continuous, one parameter family of equivalence classes. To characterize such orbits we will introduce the notion of an affine Hadamard family.

DEFINITION 3.3 An *affine Hadamard family* $H(\mathcal{R})$ stemming from a dephased $N\times N$ complex Hadamard matrix H is the set of matrices satisfying (4), associated with a subspace $\mathcal{R}$ of a space of all real $N\times N$ matrices with zeros in the first row and column,

$$H(\mathcal{R}) \ = \ \{H\circ \mathrm{EXP}(\boldsymbol{i}\cdot R)\ :\ R\in\mathcal{R}\}\,. \tag{40}$$

The words 'family' and 'orbit' denote submanifolds of $\mathbb{R}^{2n^2}$ consisting purely of dephased complex Hadamard matrices. We will often write $H(\alpha_1,\ldots,\alpha_m)$ if $\mathcal{R}$ is known to be an m-dimensional space with basis $R_1,\ldots,R_m$. In this case, $H(\alpha_1,\ldots,\alpha_m)$ will also denote the element of an affine Hadamard family:

$$H(\alpha_1,\ldots,\alpha_m) \quad = \quad H(R) \ \stackrel{\mathrm{def}}{=}\ H\circ\mathrm{EXP}(\boldsymbol{i}\cdot R) \qquad \text{where} \quad R \ = \ \alpha_1\cdot R_1+\ldots+\alpha_m\cdot R_m\,. \tag{41}$$

An affine Hadamard family $H(\mathcal{R})$ stemming from a dephased $N\times N$ complex Hadamard matrix H is called a *maximal affine Hadamard family* when it is not contained in any larger affine Hadamard family $H(\mathcal{R}')$ stemming from H,

$$\mathcal{R}\subset\mathcal{R}' \quad\Longrightarrow\quad \mathcal{R} \ = \ \mathcal{R}'\,. \tag{42}$$

Calculation of the defect of H, defined in the previous section, is a step towards determination of affine Hadamard families stemming from H:

LEMMA 3.2 *There are no affine Hadamard families stemming from a dephased $N \times N$ complex Hadamard matrix H if it is isolated, in particular if the defect of H is equal to* 0.

LEMMA 3.3 *The dimension of a continuous Hadamard orbit stemming from a dephased Hadamard matrix H is not greater than the defect $d(H)$.*

A lower bound $d_c(N)$ for the maximal dimensionality of a continuous orbit of inequivalent Hadamard matrices of size N was derived by Diţă [22]. Interestingly, for powers of a prime, $N = p^k$, this bound coincides with the defect (19) calculated at the Fourier matrix, which provides an upper bound for the dimension of an orbit stemming from F_N. Thus in this very case the problem of determining $d_c(N)$ restricted to orbits including F_N is solved and we know that the maximal affine Hadamard family stemming from F_N is not contained in any non-affine orbit of a larger dimension.

Finally, we introduce two notions of equivalence between affine Hadamard families:

DEFINITION 3.4 Two affine Hadamard families stemming from dephased $N \times N$ complex Hadamard matrices H_1 and H_2: $H_1(\mathcal{R}')$ and $H_2(\mathcal{R}'')$, associated with real matrix spaces $\mathcal{R}'$ and $\mathcal{R}''$ of the same dimension, are called *permutation equivalent* if there exist two permutation matrices P_r and P_c such that

$$H_2(\mathcal{R}'') \ = \ P_r \cdot H_1(\mathcal{R}') \cdot P_c \,, \tag{43}$$

i.e. there is one-to-one correspondence, by row and column permutation, between the elements of $H_1(\mathcal{R}')$ and $H_2(\mathcal{R}'')$.

Note that permutation matrices considered in the above definition must not shift the first row or column of a matrix.

DEFINITION 3.5 Two affine Hadamard families $H_1(\mathcal{R}')$ and $H_2(\mathcal{R}'')$ stemming from dephased $N \times N$ complex Hadamard matrices H_1 and H_2, associated with real matrix spaces $\mathcal{R}'$ and $\mathcal{R}''$ of the same dimension, are called *cognate* if

$$\forall \, B \in H_2(\mathcal{R}'') \quad \exists \, A \in H_1(\mathcal{R}') \qquad B \simeq A^T, \tag{44}$$

$$\forall \, A \in H_1(\mathcal{R}') \quad \exists \, B \in H_2(\mathcal{R}'') \qquad A \simeq B^T. \tag{45}$$

The family $H(\mathcal{R})$ is called *self-cognate* if

$$\forall \, B \in H(\mathcal{R}) \quad \exists \, A \in H(\mathcal{R}) \quad B \simeq A^T. \tag{46}$$

4. Construction of Hadamard Matrices

4.1. Same matrix size: reordering of entries and conjugation

If H is a dephased Hadamard matrix, so are its transpose H^T, the conjugated matrix $\overline{H}$ and the Hermitian transpose $H^\dagger$. It is not at all obvious, whether any pair of these is an equivalent pair. However, in some special cases it is so.

For example, the dephased forms of a Hadamard circulant matrix C and its transpose C^T are equivalent since C and C^T are always permutation equivalent (see also the remark in [21, p. 319]),

$$C^T = P^T \cdot C \cdot P\,, \tag{47}$$

where C is an $N \times N$ circulant matrix $C_{i,j} = x_{i-j \,\mathrm{mod}\, N}$ for some $x \in \mathbb{C}^N$, and $P = [e_1, e_N, e_{N-1}, \ldots, e_2]$, where e_i are the standard basis column vectors.

On the other hand, there are infinitely many examples of equivalent and inequivalent pairs of circulant Hadamard matrices C, $\overline{C}$, the same applying to their dephased forms.

Apart from transposition and conjugation, for certain dimensions there exist other matrix reorderings that preserve the Hadamard structure. Such operations that switch substructures of real Hadamard matrices to generate inequivalent matrices have recently been discussed in [58]. It is likely that these methods may be useful to get inequivalent complex Hadamard matrices. In this way only a finite number of inequivalent matrices can be obtained.

4.2. Same matrix size: linear variation of phases

Starting from a given Hadamard matrix H in the dephased form one may investigate, whether it is possible to perform infinitesimal changes of some of $(N-1)^2$ phases of the core of H to preserve unitarity. Assuming that all these phases (Φ_{kl}, from (8) $k, l = 2, \ldots, N$) vary linearly with free parameters one can find analytical form of such orbits, i.e. affine Hadamard families, stemming from e.g. Fourier matrices of composite dimensions [60].

To obtain affine Hadamard families stemming from H, one has to consider all pairs of rows of H. Now, taking the inner product of the rows in the i,j-th pair ($1 \le i < j \le N$), one gets zero as the sum of the corresponding values in the sequence

$$\left(H_{i,1} \cdot \overline{H}_{j,1}\,, \quad H_{i,2} \cdot \overline{H}_{j,2}\,, \quad \ldots\ldots\,, \quad H_{i,N} \cdot \overline{H}_{j,N}\right). \tag{48}$$

Such sequences will further be called *chains*, their subsequences — *subchains*. Thus (48) features the i,j-th chain of H. A chain (subchain) is *closed* if its elements add up to zero. As $(1/\sqrt{N}) \cdot H$ is unitary, all its chains are closed. It is not obvious, however, that any of these chains contain closed subchains.

Let us now construct a *closed subchain pattern* for H. For each chain of H, let us split it, disjointly, into closed subchains:

$$\left(H_{i,k_1^{(s)}} \cdot \overline{H}_{j,k_1^{(s)}}\,, \quad H_{i,k_2^{(s)}} \cdot \overline{H}_{j,k_2^{(s)}}\,, \quad \ldots\ldots\,, \quad H_{i,k_{p^{(s)}}^{(s)}} \cdot \overline{H}_{j,k_{p^{(s)}}^{(s)}}\right), \tag{49}$$

where $s \in \{1, 2, \ldots, r_{i,j}\}$ designates subchains, $r_{i,j}$ being the number of closed subchains the i,j-th chain is split into

$$\bigcup_{s=1}^{r_{i,j}} \left\{k_1^{(s)}, \ldots, k_{p^{(s)}}^{(s)}\right\} \;=\; \{1, 2, \ldots, N\}\,,$$

$p^{(s)}$ being the length of the s-th subchain,

$$\left\{k_1^{(s_1)}, \ldots, k_{p^{(s_1)}}^{(s_1)}\right\} \cap \left\{k_1^{(s_2)}, \ldots, k_{p^{(s_2)}}^{(s_2)}\right\} = \emptyset\,, \quad \text{if} \quad s_1 \neq s_2\,,$$

and splitting of $\{1, 2, \ldots, N\}$ into $\{k_1^{(s)}, \ldots, k_{p^{(s)}}^{(s)}\}$ is done independently for each chain of H, that is the k-values above in fact depend also on i, j.

A pattern according to which all the chains of H are split, in the above way, into closed subchains, will be called a *closed subchain pattern*.

A closed subchain pattern may give rise to the affine Hadamard family, stemming from H, corresponding to this pattern. The space $\mathcal{R}$, generating this $H(\mathcal{R})$ family (see Definition 3.3), is defined by the equations:

$$\begin{cases} R_{1,j} = 0\,, & j \in \{2, \ldots, N\}\,, \quad (50) \\ R_{i,1} = 0\,, & i \in \{1, \ldots, N\}\,, \quad (51) \end{cases}$$

(as $H(\mathcal{R})$ is made of dephased Hadamard matrices) and, for all $1 \leq i < j \leq N$ and $s \in \{1, 2, \ldots, r_{i,j}\}$,

$$\left(R_{i,k_1^{(s)}} - R_{j,k_1^{(s)}}\right) = \ldots = \left(R_{i,k_{p^{(s)}}^{(s)}} - R_{j,k_{p^{(s)}}^{(s)}}\right), \tag{52}$$

where the sets of indices $\{k_1^{(s)}, \ldots, k_{p^{(s)}}^{(s)}\}$ correspond to the considered pattern splitting of the i, j-th chain into closed subchains (49). Recall that the sets of k's depend on i, j's, ommited for simplicity of notation.

If the system (50), (51), (52) yields a nonzero space $\mathcal{R}$, then $H(\mathcal{R})$ is the affine Hadamard family corresponding to the chosen closed subchain pattern. Actually, any affine Hadamard family stemming from H is generated by some space $\mathcal{R}$ contained in a (probably larger) space $\mathcal{R}'$, corresponding to some closed subchain pattern for H. The respective theorem and its proof will be published in [60].

It may happen that the system (50), (51), (52), for pattern $\boldsymbol{P_1}^{(H)}$, defines space $\mathcal{R}$, which is also obtained as the solution to (50), (51), (52) system shaped by another pattern $\boldsymbol{P_2}^{(H)}$, imposing stronger conditions of type (52), as a result of there being longer subchains in $\boldsymbol{P_2}^{(H)}$ composed of more than one subchain of $\boldsymbol{P_1}^{(H)}$, for a given pair i, j. If this is not the case, we say that $\mathcal{R}$ is *strictly associated* with pattern $\boldsymbol{P_1}^{(H)}$. Then the subchains of the i, j-th chain of $H(R) = H \circ \mathrm{EXP}(\boldsymbol{i} \cdot R)$, $R \in \mathcal{R}$, distinguished according to pattern $\boldsymbol{P_1}^{(H)}$:

$$\Bigg(H_{i,k_1^{(s)}} \overline{H}_{j,k_1^{(s)}} \exp\left(\boldsymbol{i} \cdot (R_{i,k_1^{(s)}} - R_{j,k_1^{(s)}})\right), \\ H_{i,k_2^{(s)}} \overline{H}_{j,k_2^{(s)}} \exp\left(\boldsymbol{i} \cdot (R_{i,k_2^{(s)}} - R_{j,k_2^{(s)}})\right), \\ \ldots\ldots, \\ H_{i,k_{p^{(s)}}^{(s)}} \overline{H}_{j,k_{p^{(s)}}^{(s)}} \exp\left(\boldsymbol{i} \cdot (R_{i,k_{p^{(s)}}^{(s)}} - R_{j,k_{p^{(s)}}^{(s)}})\right)\Bigg) \tag{53}$$

rotate independently as R runs along $\mathcal{R}$ satisfying (52).

Maximal affine Hadamard families stemming from H are generated by maximal, in the sense of (42), solutions to (50), (51), (52) systems shaped by some specific closed subchain patterns, which we also call *maximal*. We have been able to find all of these for almost every complex Hadamard matrix considered in our catalogue. To our understanding, however, it becomes a serious combinatorial problem already for $N = 12$. For example, for a real 12×12 Hadamard matrix, each chain can be split in $(6!)^2$ ways into two element closed subchains only.

Fortunately, as far as Fourier matrices are concerned, the allowed maximal subchain patterns are especially regular. We are thus able to obtain all maximal affine Hadamard families stemming from F_N for an arbitrary N [60], as we have done for $N \leq 16$.

An alternative method of constructing affine Hadamard families, developed by Diţă [22], is presented in Sect. 4.5. We also refer the reader to the article by Nicoara [20], in which conditions are given for the existence of one parameter families of commuting squares of finite dimensional von Neumann algebras. These conditions can be used to establish the existence of one parameter families of complex Hadamard matrices, stemming from some H, which are not assumed to be affine.

4.3. Duplication of the matrix size

Certain ways of construction of real Hadamard matrices of an extended size work also in the complex case. If A and B belong to $\mathcal{H}_N$ then

$$H = \begin{bmatrix} A & B \\ A & -B \end{bmatrix} \in \mathcal{H}_{2N} . \tag{54}$$

Furthermore, if A and B are taken to be in the dephased form, so is H. This method, originally due to Hadamard, can be generalized by realizing that B can be multiplied at left by an arbitrary diagonal unitary matrix

$$E = \mathrm{diag}(1, e^{id_1}, \ldots, e^{id_{N-1}}) .$$

If A and B depend on a and b free parameters, respectively, then

$$H' = \begin{bmatrix} A & EB \\ A & -EB \end{bmatrix} \tag{55}$$

represents an $(a + b + N - 1)$-parameter family of Hadamard matrices of size $2N$ in the dephased form.

4.4. Quadruplication of the matrix size

In analogy to (54) one may quadruple the matrix size, in a construction similar to that derived by Williamson [61] from quaternions. If $A, B, C, D \in \mathcal{H}_N$ are in the dephased form then

$$\begin{bmatrix} A & B & C & D \\ A & -B & C & -D \\ A & B & -C & -D \\ A & -B & -C & D \end{bmatrix} \in \mathcal{H}_{4N} \tag{56}$$

and it has the dephased form. This form is preserved, if the blocks B, C and D are multiplied by diagonal unitary matrices E_1, E_2 and E_3 respectively, each containing unity and $N-1$ free phases. Therefore (56) describes an $[a+b+c+d+3(N-1)]$-dimensional family of Hadamard matrices [22], where a, b, c, d denote the number of free parameters contained in A, B, C, D, respectively.

4.5. Generalized method related to tensor product

It is not difficult to design a similar method which increases the size of a Hadamard matrix by eight, but more generally, we can use tensor product[6] to increase the size of the matrix K times. For any two Hadamard matrices, $A \in \mathcal{H}_K$ and $B \in \mathcal{H}_M$, their tensor product $A \otimes B \in \mathcal{H}_{KM}$. A more general construction by Diţă allows to use entire set of K (possibly different) Hadamard matrices $\{B_1, \ldots, B_K\}$ of size M. Then the matrix

$$H = \begin{bmatrix} A_{11}B_1 & A_{12}E_2B_2 & . & . & A_{1K}E_KB_K \\ . & . & . & . & . \\ . & . & . & . & . \\ A_{K1}B_1 & A_{K2}E_2B_2 & . & . & A_{KK}E_KB_K \end{bmatrix} \tag{57}$$

of size $N = KM$ is Hadamard [22]. As before we introduce additional free phases by using $K-1$ diagonal unitary matrices E_k. Each matrix depends on $M-1$ phases, since the constraint $[E_k]_{1,1} = 1$ for $k = 2, \ldots, K$ is necessary to preserve the dephased form of H. Thus this orbit of (mostly) not equivalent Hadamard matrices depends on

$$d = a + \sum_{j=1}^{K} b_j + (K-1)(M-1) \tag{58}$$

free parameters. Here a denotes the number of free parameters in A, while b_j denotes the number of free parameters in B_j. A similar construction giving at least $(K-1)(M-1)$ free parameters was given by Haagerup [21]. In the simplest case $K \cdot M = 2 \cdot 2$ these methods give the standard 1-parameter $N = 4$ family (65), while for $K \cdot M = 2 \cdot 3$ one arrives with $(2-1)(3-1) = 2$ parameter family (69) of (mostly) inequivalent $N = 6$ Hadamard matrices.

The tensor product construction can work only for composite N, so it was conjectured [29] that for a prime dimension N there exist only finitely many inequivalent complex Hadamard matrices. However, this occurred to be false after a discovery by Petrescu [62], who found continuous families of complex Hadamard

[6] Tensor products of Hadamard matrices of the Butson type were investigated in [39].

matrices for certain prime dimensions. We are going to present his solution for $N = 7$ and $N = 13$, while a similar construction [62] works also for $N = 19, 31$ and 79. For all primes $N \geq 7$ there exist at least three isolated complex Hadamard matrices, see [21, Thm. 3.10, p. 320].

5. Catalogue of Complex Hadamard Matrices

In this section, we list complex Hadamard matrices known to us. To save space we do not describe their construction but present a short characterization of each case.

Each entry H, provided in a dephased form, represents a continuous $(2N-1)$-dimensional family of matrices $\boldsymbol{H}$ obtained by multiplication by diagonal unitary matrices:

$$\boldsymbol{H}(\alpha_1, \ldots, \alpha_N, \beta_2, \ldots, \beta_N) = D_1(\alpha_1, \ldots, \alpha_N) \cdot H \cdot D_2(\beta_2, \ldots, \beta_N)\,, \tag{59}$$

where

$$\begin{aligned} D_1(\alpha_1, \ldots, \alpha_N) &= \operatorname{diag}(e^{i\alpha_1}, \ldots, e^{i\alpha_N})\,, \\ D_2(\beta_2, \ldots, \beta_N) &= \operatorname{diag}(1, e^{i\beta_2}, \ldots, e^{i\beta_N})\,. \end{aligned}$$

Furthermore, each H in the list represents matrices obtained by discrete permutations of its rows and columns, and equivalent in the sense of (5). For a given size N we enumerate families by capital letters associated with a given construction. The superscript in brackets denotes the dimension of an orbit. For instance, $F_6^{(2)}$ represents the two-parameter family of $N = 6$ complex Hadamard matrices stemming from the Fourier matrix F_6. Displaying continuous families of Hadamard matrices we shall use the symbol $\bullet$ to denote zeros in phase variation matrices. For completeness we shall start with a trivial case.

5.1. $N = 1$

$$F_1^{(0)} = F_1 = [1]\,.$$

5.2. $N = 2$

All complex 2×2 Hadamard matrices are equivalent to the Fourier matrix F_2 [21, Prop. 2.1, p. 298],

$$F_2 = F_2^{(0)} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \tag{60}$$

The complex Hadamard matrix F_2 is isolated. The set of inequivalent Hadamard matrices of size 2 contains one element only, $\mathcal{G}_2 = \{F_2\}$.

5.3. $N = 3$

All complex 3×3 Hadamard matrices are equivalent to the Fourier matrix F_3 [21, Prop. 2.1, p. 298],

$$F_3 \;=\; F_3^{(0)} \;=\; \begin{bmatrix} 1 & 1 & 1 \\ 1 & w & w^2 \\ 1 & w^2 & w \end{bmatrix}, \tag{61}$$

where $w = \exp(\boldsymbol{i}\cdot 2\pi/3)$, so $w^3 = 1$. F_3 is an isolated complex Hadamard matrix. The set of dephased representatives can be taken as $\mathcal{G}_3 = \{F_3\}$.

5.4. $N = 4$

Every 4×4 complex Hadamard matrix is equivalent to a matrix belonging to the only maximal affine Hadamard family $F_4^{(1)}(a)$ stemming from F_4 [21]. The $F_4^{(1)}(a)$ family is given by the formula

$$F_4^{(1)}(a) \;=\; F_4 \circ \mathrm{EXP}\Big(\boldsymbol{i}\cdot R_{F_4^{(1)}}(a)\Big), \tag{62}$$

where

$$F_4 \;=\; \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & w & w^2 & w^3 \\ 1 & w^2 & 1 & w^2 \\ 1 & w^3 & w^2 & w \end{bmatrix} \;=\; \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \boldsymbol{i} & -1 & -\boldsymbol{i} \\ 1 & -1 & 1 & -1 \\ 1 & -\boldsymbol{i} & -1 & \boldsymbol{i} \end{bmatrix}, \tag{63}$$

with $w = \exp(\boldsymbol{i}\cdot 2\pi/4) = \boldsymbol{i}$, so $w^4 = 1$, $w^2 = -1$, and

$$R_{F_4^{(1)}}(a) \;=\; \left[\begin{array}{cc|cc} \bullet & \bullet & \bullet & \bullet \\ \bullet & a & \bullet & a \\ \hline \bullet & \bullet & \bullet & \bullet \\ \bullet & a & \bullet & a \end{array}\right]. \tag{64}$$

Thus the orbit stemming from F_4 reads

$$F_4^{(1)}(a) \;=\; \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \boldsymbol{i}\cdot e^{\boldsymbol{i}a} & -1 & -\boldsymbol{i}\cdot e^{\boldsymbol{i}a} \\ 1 & -1 & 1 & -1 \\ 1 & -\boldsymbol{i}\cdot e^{\boldsymbol{i}a} & -1 & \boldsymbol{i}\cdot e^{\boldsymbol{i}a} \end{bmatrix}. \tag{65}$$

The above orbit is permutation equivalent to

$$\widetilde{F}_4^{(1)}(\alpha) \;=\; \left[\begin{array}{c|c} [F_2]_{1,1}\cdot F_2 & [F_2]_{1,2}\cdot D(\alpha)\cdot F_2 \\ \hline [F_2]_{2,1}\cdot F_2 & [F_2]_{2,2}\cdot D(\alpha)\cdot F_2 \end{array}\right], \tag{66}$$

where $D(\alpha)$ is the 2×2 diagonal matrix $\mathrm{diag}(1, e^{\boldsymbol{i}\alpha})$.

This orbit is constructed with the Diţă's method [22], by setting $K = M = 2$, $A = B_1 = B_2 = F_2$, $E_2 = D(\alpha)$ in (57). It passes through a permuted F_4:

$$F_4 \;=\; \widetilde{F}_4^{(1)}(\pi/2)\cdot[e_1, e_3, e_2, e_4]^T, \tag{67}$$

where e_i is the i-th standard basis column vector, and through $F_2 \otimes F_2 = \widetilde{F}_4^{(1)}(0)$. Note that F_4 and $F_2 \otimes F_2$ are, according to [59], inequivalent.

The $F_4^{(1)}$ orbit of (62) is symmetric, so it is self-cognate. Replacing a by $a+\pi$ yields $F_4^{(1)}(a)$ with the 2-nd and 4-th column exchanged, so $F_4^{(1)}(a) \simeq F_4^{(1)}(a+\pi)$. The set of dephased representatives can be taken as $\mathcal{G}_4 = \{F_4^{(1)}(a) : \ a \in [0,\pi)\}$.

5.5. $N = 5$

As shown by Haagerup in [21, Th. 2.2, p. 298] for $N = 5$ all complex Hadamard matrices are equivalent to the Fourier matrix F_5,

$$F_5 \ = \ F_5^{(0)} \ = \ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & w & w^2 & w^3 & w^4 \\ 1 & w^2 & w^4 & w & w^3 \\ 1 & w^3 & w & w^4 & w^2 \\ 1 & w^4 & w^3 & w^2 & w \end{bmatrix}, \tag{68}$$

where $w = \exp(\boldsymbol{i} \cdot 2\pi/5)$ so $w^5 = 1$. The above matrix F_5 is isolated. The set of dephased representatives can be taken as $\mathcal{G}_5 = \{F_5\}$.

5.6. $N = 6$

5.6.1. Orbits stemming from F_6

The only maximal affine Hadamard families stemming from F_6 are

$$F_6^{(2)}(a,b) \ = \ F_6 \circ \mathrm{EXP}\left(\boldsymbol{i} \cdot R_{F_6^{(2)}}(a,b)\right), \tag{69}$$

$$\left(F_6^{(2)}(a,b)\right)^T \ = \ F_6 \circ \mathrm{EXP}\left(\boldsymbol{i} \cdot \left(R_{F_6^{(2)}}(a,b)\right)^T\right), \tag{70}$$

where

$$F_6 \ = \ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & w & w^2 & w^3 & w^4 & w^5 \\ 1 & w^2 & w^4 & 1 & w^2 & w^4 \\ 1 & w^3 & 1 & w^3 & 1 & w^3 \\ 1 & w^4 & w^2 & 1 & w^4 & w^2 \\ 1 & w^5 & w^4 & w^3 & w^2 & w \end{bmatrix}, \tag{71}$$

with $w = \exp(\boldsymbol{i} \cdot 2\pi/6)$ so $w^6 = 1$, $w^3 = -1$, and

$$R_{F_6^{(2)}}(a,b) \ = \ \left[\begin{array}{ccc|ccc} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & a & b & \bullet & a & b \\ \hline \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & a & b & \bullet & a & b \\ \hline \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & a & b & \bullet & a & b \end{array}\right]. \tag{72}$$

The above families (69) and (70) are cognate. The defect (see Def. 3.2) reads $d(F_6) = 4$.

At least one of the above orbits is permutation equivalent to one of the orbits obtained using the Diţă's method: either

$$\widetilde{F}_{6A}^{(2)}(\alpha_1,\alpha_2) = \left[\begin{array}{c|c|c} [F_3]_{1,1}\cdot F_2 & [F_3]_{1,2}\cdot D(\alpha_1)\cdot F_2 & [F_3]_{1,3}\cdot D(\alpha_2)\cdot F_2 \\ \hline [F_3]_{2,1}\cdot F_2 & [F_3]_{2,2}\cdot D(\alpha_1)\cdot F_2 & [F_3]_{2,3}\cdot D(\alpha_2)\cdot F_2 \\ \hline [F_3]_{3,1}\cdot F_2 & [F_3]_{3,2}\cdot D(\alpha_1)\cdot F_2 & [F_3]_{3,3}\cdot D(\alpha_2)\cdot F_2 \end{array}\right], \tag{73}$$

where $D(\alpha)$ is the 2×2 diagonal matrix $\mathrm{diag}(1, e^{\boldsymbol{i}\alpha})$, by setting $K=3$, $M=2$, $A=F_3$, $B_1=B_2=B_3=F_2$, $E_2=D(\alpha_1)$, $E_3=D(\alpha_2)$ in (57), or

$$\widetilde{F}_{6B}^{(2)}(\beta_1,\beta_2) = \left[\begin{array}{c|c} [F_2]_{1,1}\cdot F_3 & [F_2]_{1,2}\cdot D(\beta_1,\beta_2)\cdot F_3 \\ \hline [F_2]_{2,1}\cdot F_3 & [F_2]_{2,2}\cdot D(\beta_1,\beta_2)\cdot F_3 \end{array}\right], \tag{74}$$

where $D(\beta_1,\beta_2)$ is the 3×3 diagonal matrix $\mathrm{diag}(1, e^{\boldsymbol{i}\beta_1}, e^{\boldsymbol{i}\beta_2})$, by setting $K=2$, $M=3$, $A=F_2$, $B_1=B_2=F_3$, $E_2=D(\beta_1,\beta_2)$ in (57).

The above statement is true because the orbits (73) and (74) pass through $\widetilde{F}_{6A}^{(2)}(0,0)=F_3\otimes F_2$ and $\widetilde{F}_{6B}^{(2)}(0,0)=F_2\otimes F_3$, both of which are, according to [59], permutation equivalent to F_6. Thus (73) and (74) are maximal affine Hadamard families stemming from permuted F_6's. These orbits were constructed in the work of Haagerup [21].

A change of the phase a by π in the family $F_6^{(2)}$ corresponds to the exchange of the 2-nd and 5-th column of $F_6^{(2)}(a,b)$, while a change of b by π is equivalent to the exchange of the 3-rd and 6-th column of $F_6^{(2)}(a,b)$. This implies the (permutation) equivalence relation

$$F_6^{(2)}(a,b) \simeq F_6^{(2)}(a+\pi,b) \simeq F_6^{(2)}(a,b+\pi) \simeq F_6^{(2)}(a+\pi,b+\pi). \tag{75}$$

5.6.2. 1-parameter orbits

There are precisely five permutation equivalent 1-parameter maximal affine Hadamard families stemming from the symmetric matrix D_6,

$$D_6 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \boldsymbol{i} & -\boldsymbol{i} & -\boldsymbol{i} & \boldsymbol{i} \\ 1 & \boldsymbol{i} & -1 & \boldsymbol{i} & -\boldsymbol{i} & -\boldsymbol{i} \\ 1 & -\boldsymbol{i} & \boldsymbol{i} & -1 & \boldsymbol{i} & -\boldsymbol{i} \\ 1 & -\boldsymbol{i} & -\boldsymbol{i} & \boldsymbol{i} & -1 & \boldsymbol{i} \\ 1 & \boldsymbol{i} & -\boldsymbol{i} & -\boldsymbol{i} & \boldsymbol{i} & -1 \end{bmatrix}, \tag{76}$$

namely

$$D_6^{(1)}(c) = D_6 \circ \mathrm{EXP}\left(\boldsymbol{i}\cdot R_{D_6^{(1)}}(c)\right), \tag{77}$$

where

$$R_{D_6^{(1)}}(c) = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & c & c & \bullet \\ \bullet & \bullet & -c & \bullet & \bullet & -c \\ \bullet & \bullet & -c & \bullet & \bullet & -c \\ \bullet & \bullet & \bullet & c & c & \bullet \end{bmatrix} \tag{78}$$

and

$$P_1 \cdot D_6^{(1)}(c) \cdot P_1^T = D_6 \circ \mathrm{EXP}\left(\boldsymbol{i} \cdot P_1 R_{D_6^{(1)}}(c) P_1^T\right), \tag{79}$$

$$P_2 \cdot D_6^{(1)}(c) \cdot P_2^T = D_6 \circ \mathrm{EXP}\left(\boldsymbol{i} \cdot P_2 R_{D_6^{(1)}}(c) P_2^T\right), \tag{80}$$

$$P_3 \cdot D_6^{(1)}(c) \cdot P_3^T = D_6 \circ \mathrm{EXP}\left(\boldsymbol{i} \cdot P_3 R_{D_6^{(1)}}(c) P_3^T\right), \tag{81}$$

$$P_4 \cdot D_6^{(1)}(c) \cdot P_4^T = D_6 \circ \mathrm{EXP}\left(\boldsymbol{i} \cdot P_4 R_{D_6^{(1)}}(c) P_4^T\right), \tag{82}$$

where

$$P_1 = [e_1, e_3, e_2, e_6, e_5, e_4], \tag{83}$$

$$P_2 = [e_1, e_4, e_3, e_2, e_6, e_5], \tag{84}$$

$$P_3 = [e_1, e_6, e_2, e_3, e_4, e_5], \tag{85}$$

$$P_4 = [e_1, e_5, e_4, e_3, e_2, e_6] \tag{86}$$

and e_i denotes the i-th standard basis column vector.

We have the permutation equivalence

$$D_6^{(1)}(c+\pi) = P_-^T \cdot D_6^{(1)}(c) \cdot P_- \quad \text{for} \quad P_- = [e_1, e_2, e_3, e_5, e_4, e_6]. \tag{87}$$

Also

$$D_6^{(1)}(-c) = \left(D_6^{(1)}(c)\right)^T, \tag{88}$$

thus

$$\left\{D_6^{(1)}(c) : c \in [0, 2\pi)\right\} \quad \text{and} \quad \left\{\left(D_6^{(1)}(c)\right)^T : c \in [0, 2\pi)\right\}$$

are equal sets, that is $D_6^{(1)}$ is a self-cognate family. None of the matrices of $F_6^{(2)}$ and $(F_6^{(2)})^T$ of (69) and (70) are equivalent to any of the matrices of $D_6^{(1)}$. Obviously, the above remarks apply to the remaining orbits stemming from D_6.

The $D_6^{(1)}$ orbit was presented in [22] in the Introduction, and its 'starting point' matrix D_6 of (76) even in the earlier work [21, p. 307] (not dephased).

5.6.3. The 'cyclic 6 roots' matrix

There exists another, inequivalent to any of the above 6×6 matrices, complex Hadamard matrix derived in [21] from the results of [32] on so-called cyclic 6 roots.

The matrix $\widetilde{C}_6^{(0)}$ below is circulant, i.e. it has the structure

$$\left[\widetilde{C}_6^{(0)}\right]_{i,j} = x_{(i-j \bmod 6)+1}, \quad \text{where} \quad x = [1,\ \boldsymbol{i}/d,\ -1/d,\ -\boldsymbol{i},\ -d,\ \boldsymbol{i}d]^T \tag{89}$$

and

$$d = \frac{1-\sqrt{3}}{2} + \boldsymbol{i} \cdot \left(\frac{\sqrt{3}}{2}\right)^{1/2} \tag{90}$$

is the root of the equation $d^2-(1-\sqrt{3})d+1=0$.

The matrix $\widetilde{C}_6^{(0)}$ and its dephased form $C_6^{(0)}$ read

$$\widetilde{C}_6^{(0)} = \begin{bmatrix} 1 & \boldsymbol{i}\,d & -d & -\boldsymbol{i} & -d^{-1} & \boldsymbol{i}d^{-1} \\ \boldsymbol{i}d^{-1} & 1 & \boldsymbol{i}\,d & -d & -\boldsymbol{i} & -d^{-1} \\ -d^{-1} & \boldsymbol{i}d^{-1} & 1 & \boldsymbol{i}\,d & -d & -\boldsymbol{i} \\ -\boldsymbol{i} & -d^{-1} & \boldsymbol{i}d^{-1} & 1 & \boldsymbol{i}\,d & -d \\ -d & -\boldsymbol{i} & -d^{-1} & \boldsymbol{i}d^{-1} & 1 & \boldsymbol{i}\,d \\ \boldsymbol{i}\,d & -d & -\boldsymbol{i} & -d^{-1} & \boldsymbol{i}d^{-1} & 1 \end{bmatrix} \tag{91}$$

$$C_6^{(0)} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -d & -d^2 & d^2 & d \\ 1 & -d^{-1} & 1 & d^2 & -d^3 & d^2 \\ 1 & -d^{-2} & d^{-2} & -1 & d^2 & -d^2 \\ 1 & d^{-2} & -d^{-3} & d^{-2} & 1 & -d \\ 1 & d^{-1} & d^{-2} & -d^{-2} & -d^{-1} & -1 \end{bmatrix}. \tag{92}$$

The circulant structure of $\widetilde{C}_6^{(0)}$ implies that it is equivalent to $(\widetilde{C}_6^{(0)})^T$, so $C_6^{(0)} \simeq (C_6^{(0)})^T$ (see (47)). Thus $C_6^{(0)} \simeq \overline{C_6^{(0)}}$ ($\Longleftrightarrow \widetilde{C}_6^{(0)} \simeq \overline{\widetilde{C}_6^{(0)}}$).

No affine Hadamard family stems from $C_6^{(0)}$. However, we do not know if the cyclic-6-roots matrix is isolated since the defect $d(C_6^{(0)})=4$ and we cannot exclude existence of some other orbit.

5.6.4. The 'spectral set' 6×6 *matrix*

Another complex Hadamard matrix found by Tao [35] plays an important role in the investigation of spectral sets and disproving the Fuglede's conjecture [36, 38]. It is a symmetric matrix $S_6^{(0)}$, which belongs to the Butson class $H(3,6)$, so its entries depend on the third root of unity $\omega=\exp(\boldsymbol{i}\cdot 2\pi/3)$,

$$S_6^{(0)} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & \omega & \omega & \omega^2 & \omega^2 \\ 1 & \omega & 1 & \omega^2 & \omega^2 & \omega \\ 1 & \omega & \omega^2 & 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega^2 & \omega & 1 & \omega \\ 1 & \omega^2 & \omega & \omega^2 & \omega & 1 \end{bmatrix}. \tag{93}$$

Thus the corresponding log-Hadamard matrix reads

$$\Phi_{S_6} = \frac{2\pi}{3}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 & 2 \\ 0 & 1 & 0 & 2 & 2 & 1 \\ 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 2 & 2 & 1 & 0 & 1 \\ 0 & 2 & 1 & 2 & 1 & 0 \end{bmatrix}. \tag{94}$$

Its defect is equal to zero, $d(S_6)=0$, hence the matrix (93) is isolated. Spectral sets allow to construct certain Hadamard matrices for other composite dimensions (see [37, Prop. 2.2.]), but it is not yet established, in which cases this method yields new solutions.

5.7. $N = 7$

5.7.1. *Orbits stemming from* F_7

F_7 is an isolated 7×7 complex Hadamard matrix,

$$F_7 \;=\; F_7^{(0)} \;=\; \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & w & w^2 & w^3 & w^4 & w^5 & w^6 \\ 1 & w^2 & w^4 & w^6 & w & w^3 & w^5 \\ 1 & w^3 & w^6 & w^2 & w^5 & w & w^4 \\ 1 & w^4 & w & w^5 & w^2 & w^6 & w^3 \\ 1 & w^5 & w^3 & w & w^6 & w^4 & w^2 \\ 1 & w^6 & w^5 & w^4 & w^3 & w^2 & w \end{bmatrix}, \tag{95}$$

where $w = \exp(\boldsymbol{i} \cdot 2\pi/7)$, so $w^7 = 1$.

5.7.2. 1-*parameter orbits*

There are precisely three permutation equivalent 1-parameter maximal affine Hadamard families stemming from the symmetric matrix being a permuted 'starting point' for the 1-parameter orbit found by Petrescu [62],

$$P_7 \;=\; \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & w & w^4 & w^5 & w^3 & w^3 & w \\ 1 & w^4 & w & w^3 & w^5 & w^3 & w \\ 1 & w^5 & w^3 & w & w^4 & w & w^3 \\ 1 & w^3 & w^5 & w^4 & w & w & w^3 \\ 1 & w^3 & w^3 & w & w & w^4 & w^5 \\ 1 & w & w & w^3 & w^3 & w^5 & w^4 \end{bmatrix}, \tag{96}$$

where $w = \exp(\boldsymbol{i} \cdot 2\pi/6)$, so $w^6 = 1,\ w^3 = -1$. They are

$$P_7^{(1)}(a) \;=\; P_7 \circ \mathrm{EXP}\left(\boldsymbol{i} \cdot R_{P_7^{(1)}}(a)\right), \tag{97}$$

where

$$R_{P_7^{(1)}}(a) \;=\; \begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & a & a & \bullet & \bullet & \bullet & \bullet \\ \bullet & a & a & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & -a & -a & \bullet & \bullet \\ \bullet & \bullet & \bullet & -a & -a & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \end{bmatrix} \tag{98}$$

and

$$P_1 \cdot P_7^{(1)}(a) \cdot P_2^T \;=\; P_7 \circ \mathrm{EXP}\left(\boldsymbol{i} \cdot P_1 R_{P_7^{(1)}}(a) P_2^T\right), \tag{99}$$

$$P_2 \cdot P_7^{(1)}(a) \cdot P_1^T \;=\; P_7 \circ \mathrm{EXP}\left(\boldsymbol{i} \cdot P_2 R_{P_7^{(1)}}(a) P_1^T\right), \tag{100}$$

where

$$P_1 \;=\; [e_1,\ e_7,\ e_3,\ e_4,\ e_6,\ e_5,\ e_2]\,, \quad P_2 \;=\; [e_1,\ e_2,\ e_7,\ e_6,\ e_5,\ e_4,\ e_3]\,, \tag{101}$$

and e_i denotes the i-th standard basis column vector.

The above orbits are permutation equivalent to the 7×7 family found by Petrescu [62]. They all are cognate, and the $P_7^{(1)}$ orbit is even self-cognate since

$$P_7^{(1)}(a) \;=\; \left(P_7^{(1)}(a)\right)^T . \tag{102}$$

Also

$$P_7^{(1)}(-a) \;=\; P^T \cdot P_7^{(1)}(a) \cdot P \quad \text{for} \quad P \;=\; [e_1,\, e_4,\, e_5,\, e_2,\, e_3,\, e_7,\, e_6]\,, \tag{103}$$

so $P_7^{(1)}(-a) \simeq P_7^{(1)}(a)$

Due to some freedom in the construction of family components the method of Petrescu allows one to build other families of Hadamard matrices similar to $P_7^{(1)}$. Not knowing if they are inequivalent we are not going to consider them here.

5.7.3. The 'cyclic 7 roots' matrices

There exist only four inequivalent 7×7 complex Hadamard matrices, inequivalent to F_7 and the 1-parameter family found by Petrescu (see previous subsection), associated with nonclassical cyclic 7 roots. This result was obtained in [21] and is based on the catalogue of all cyclic 7 roots presented in [32].

The two matrices $\widetilde{C}_{7A}^{(0)}$, $\widetilde{C}_{7B}^{(0)}$ correspond to the so-called 'index 2' solutions to the cyclic 7 roots problem. They have the circulant structure $[U]_{i,j} = x_{(i-j \bmod 7)+1}$, where

$$x \;=\; [1,\ 1,\ 1,\ d,\ 1,\ d,\ d] \quad \text{for} \quad \widetilde{C}_{7A}^{(0)}\,, \tag{104}$$

$$x \;=\; [1,\ 1,\ 1,\ \overline{d},\ 1,\ \overline{d},\ \overline{d}] \quad \text{for} \quad \widetilde{C}_{7B}^{(0)} \tag{105}$$

and

$$d \;=\; \frac{-3 + \boldsymbol{i}\sqrt{7}}{4} \quad \text{such that} \quad d^2 + \frac{3}{2}d + 1 = 0 \implies d \cdot \overline{d} = 1\,. \tag{106}$$

The corresponding dephased matrices are denoted as $C_{7A}^{(0)}$ and $C_{7B}^{(0)}$:

$$\widetilde{C}_{7A}^{(0)} \;=\; \begin{bmatrix} 1 & d & d & 1 & d & 1 & 1 \\ 1 & 1 & d & d & 1 & d & 1 \\ 1 & 1 & 1 & d & d & 1 & d \\ d & 1 & 1 & 1 & d & d & 1 \\ 1 & d & 1 & 1 & 1 & d & d \\ d & 1 & d & 1 & 1 & 1 & d \\ d & d & 1 & d & 1 & 1 & 1 \end{bmatrix} \quad C_{7A}^{(0)} \;=\; \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & d^{-1} & 1 & d & d^{-1} & d & 1 \\ 1 & d^{-1} & d^{-1} & d & 1 & 1 & d \\ 1 & d^{-2} & d^{-2} & d^{-1} & d^{-1} & 1 & d^{-1} \\ 1 & 1 & d^{-1} & 1 & d^{-1} & d & d \\ 1 & d^{-2} & d^{-1} & d^{-1} & d^{-2} & d^{-1} & 1 \\ 1 & d^{-1} & d^{-2} & 1 & d^{-2} & d^{-1} & d^{-1} \end{bmatrix} \tag{107}$$

$$\widetilde{C}_{7B}^{(0)} = \begin{bmatrix} 1 & d^{-1} & d^{-1} & 1 & d^{-1} & 1 & 1 \\ 1 & 1 & d^{-1} & d^{-1} & 1 & d^{-1} & 1 \\ 1 & 1 & 1 & d^{-1} & d^{-1} & 1 & d^{-1} \\ d^{-1} & 1 & 1 & 1 & d^{-1} & d^{-1} & 1 \\ 1 & d^{-1} & 1 & 1 & 1 & d^{-1} & d^{-1} \\ d^{-1} & 1 & d^{-1} & 1 & 1 & 1 & d^{-1} \\ d^{-1} & d^{-1} & 1 & d^{-1} & 1 & 1 & 1 \end{bmatrix} \quad C_{7B}^{(0)} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & d & 1 & d^{-1} & d & d^{-1} & 1 \\ 1 & d & d & d^{-1} & 1 & 1 & d^{-1} \\ 1 & d^2 & d^2 & d & d & 1 & d \\ 1 & 1 & d & 1 & d & d^{-1} & d^{-1} \\ 1 & d^2 & d & d & d^2 & d & 1 \\ 1 & d & d^2 & 1 & d^2 & d & d \end{bmatrix} \tag{108}$$

There holds $C_{7B}^{(0)} = \overline{C_{7A}^{(0)}}$ ($\Longleftrightarrow \widetilde{C}_{7B}^{(0)} = \overline{\widetilde{C}_{7A}^{(0)}}$) and the matrices $C_{7A}^{(0)}$, $\widetilde{C}_{7A}^{(0)}$ and $C_{7B}^{(0)}$, $\widetilde{C}_{7B}^{(0)}$ are equivalent to $(C_{7A}^{(0)})^T$, $(\widetilde{C}_{7A}^{(0)})^T$ and $(C_{7B}^{(0)})^T$, $(\widetilde{C}_{7B}^{(0)})^T$ respectively (see (47)).

The structure of $\widetilde{C}_{7C}^{(0)}, \widetilde{C}_{7D}^{(0)}$ related to 'index 3' solutions to the cyclic 7 roots problem is again $[U]_{i,j} = x_{(i-j \bmod 7)+1}$, where x is a bit more complicated. We put:

$$x = [1,\ A,\ B,\ C,\ C,\ B,\ A]^T \quad \text{for} \quad \widetilde{C}_{7C}^{(0)}, \tag{109}$$

$$x = [1,\ \overline{A},\ \overline{B},\ \overline{C},\ \overline{C},\ \overline{B},\ \overline{A}]^T \quad \text{for} \quad \widetilde{C}_{7D}^{(0)}, \tag{110}$$

where $A = a$, $B = ab$, $C = abc$ are products of algebraic numbers a, b, c of modulus equal to 1 (see the final remark 3.11 of [21]). Their numerical approximations are given by

$$a \approx \exp(\boldsymbol{i} \cdot 4.312839) \tag{111}$$

$$b \approx \exp(\boldsymbol{i} \cdot 1.356228) \tag{112}$$

$$c \approx \exp(\boldsymbol{i} \cdot 1.900668) \tag{113}$$

$$\tag{114}$$

and then

$$A = a \approx (-0.389004) + \boldsymbol{i} \cdot (-0.921236) \tag{115}$$

$$B = ab \approx (0.817282) + \boldsymbol{i} \cdot (-0.576238) \tag{116}$$

$$C = abc \approx (0.280434) + \boldsymbol{i} \cdot (0.959873) \tag{117}$$

Again, $C_{7C}^{(0)}$, $C_{7D}^{(0)}$ denote the dephased versions of $\widetilde{C}_{7C}^{(0)}$, $\widetilde{C}_{7D}^{(0)}$, which read:

$$\widetilde{C}_{7C}^{(0)} = \begin{bmatrix} 1 & A & B & C & C & B & A \\ A & 1 & A & B & C & C & B \\ B & A & 1 & A & B & C & C \\ C & B & A & 1 & A & B & C \\ C & C & B & A & 1 & A & B \\ B & C & C & B & A & 1 & A \\ A & B & C & C & B & A & 1 \end{bmatrix} \quad \widetilde{C}_{7D}^{(0)} = \begin{bmatrix} 1 & A^{-1} & B^{-1} & C^{-1} & C^{-1} & B^{-1} & A^{-1} \\ A^{-1} & 1 & A^{-1} & B^{-1} & C^{-1} & C^{-1} & B^{-1} \\ B^{-1} & A^{-1} & 1 & A^{-1} & B^{-1} & C^{-1} & C^{-1} \\ C^{-1} & B^{-1} & A^{-1} & 1 & A^{-1} & B^{-1} & C^{-1} \\ C^{-1} & C^{-1} & B^{-1} & A^{-1} & 1 & A^{-1} & B^{-1} \\ B^{-1} & C^{-1} & C^{-1} & B^{-1} & A^{-1} & 1 & A^{-1} \\ A^{-1} & B^{-1} & C^{-1} & C^{-1} & B^{-1} & A^{-1} & 1 \end{bmatrix} \tag{118}$$

Then

$$C_{7C}^{(0)} = \left[\begin{array}{c|c|c|c|c|c|c} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & A^{-2} & B^{-1} & A^{-1}BC^{-1} & A^{-1} & A^{-1}B^{-1}C & A^{-2}B \\ \hline 1 & B^{-1} & B^{-2} & AB^{-1}C^{-1} & C^{-1} & B^{-2}C & A^{-1}B^{-1}C \\ \hline 1 & A^{-1}BC^{-1} & AB^{-1}C^{-1} & C^{-2} & AC^{-2} & C^{-1} & A^{-1} \\ \hline 1 & A^{-1} & C^{-1} & AC^{-2} & C^{-2} & AB^{-1}C^{-1} & A^{-1}BC^{-1} \\ \hline 1 & A^{-1}B^{-1}C & B^{-2}C & C^{-1} & AB^{-1}C^{-1} & B^{-2} & B^{-1} \\ \hline 1 & A^{-2}B & A^{-1}B^{-1}C & A^{-1} & A^{-1}BC^{-1} & B^{-1} & A^{-2} \end{array}\right]$$

$$= \left[\begin{array}{c|c|c|c|c|c|c} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & a^{-2} & a^{-1}b^{-1} & a^{-1}c^{-1} & a^{-1} & a^{-1}c & a^{-1}b \\ \hline 1 & a^{-1}b^{-1} & a^{-2}b^{-2} & a^{-1}b^{-2}c^{-1} & a^{-1}b^{-1}c^{-1} & a^{-1}b^{-1}c & a^{-1}c \\ \hline 1 & a^{-1}c^{-1} & a^{-1}b^{-2}c^{-1} & a^{-2}b^{-2}c^{-2} & a^{-1}b^{-2}c^{-2} & a^{-1}b^{-1}c^{-1} & a^{-1} \\ \hline 1 & a^{-1} & a^{-1}b^{-1}c^{-1} & a^{-1}b^{-2}c^{-2} & a^{-2}b^{-2}c^{-2} & a^{-1}b^{-2}c^{-1} & a^{-1}c^{-1} \\ \hline 1 & a^{-1}c & a^{-1}b^{-1}c & a^{-1}b^{-1}c^{-1} & a^{-1}b^{-2}c^{-1} & a^{-2}b^{-2} & a^{-1}b^{-1} \\ \hline 1 & a^{-1}b & a^{-1}c & a^{-1} & a^{-1}c^{-1} & a^{-1}b^{-1} & a^{-2} \end{array}\right] \tag{119}$$

and

$$
C_{7D}^{(0)} = \left[\begin{array}{c|c|c|c|c|c|c}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline
1 & A^2 & B & AB^{-1}C & A & ABC^{-1} & A^2B^{-1} \\ \hline
1 & B & B^2 & A^{-1}BC & C & B^2C^{-1} & ABC^{-1} \\ \hline
1 & AB^{-1}C & A^{-1}BC & C^2 & A^{-1}C^2 & C & A \\ \hline
1 & A & C & A^{-1}C^2 & C^2 & A^{-1}BC & AB^{-1}C \\ \hline
1 & ABC^{-1} & B^2C^{-1} & C & A^{-1}BC & B^2 & B \\ \hline
1 & A^2B^{-1} & ABC^{-1} & A & AB^{-1}C & B & A^2
\end{array}\right]
= \left[\begin{array}{c|c|c|c|c|c|c}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline
1 & a^2 & ab & ac & a & ac^{-1} & ab^{-1} \\ \hline
1 & ab & a^2b^2 & ab^2c & abc & abc^{-1} & ac^{-1} \\ \hline
1 & ac & ab^2c & a^2b^2c^2 & ab^2c^2 & abc & a \\ \hline
1 & a & abc & ab^2c^2 & a^2b^2c^2 & ab^2c & ac \\ \hline
1 & ac^{-1} & abc^{-1} & abc & ab^2c & a^2b^2 & ab \\ \hline
1 & ab^{-1} & ac^{-1} & a & ac & ab & a^2
\end{array}\right] \tag{120}
$$

The matrices $C_{7C}^{(0)}$ and $C_{7D}^{(0)}$ are symmetric and are related by complex conjugation, $C_{7D}^{(0)} = \overline{C_{7C}^{(0)}}$. All four cyclic 7-root Hadamard matrices are isolated [20].

5.8. $N = 8$

5.8.1. Orbits stemming from F_8

The only maximal affine Hadamard family stemming from F_8 is the 5-parameter orbit,

$$
F_8^{(5)}(a,b,c,d,e) = F_8 \circ \mathrm{EXP}(\boldsymbol{i} \cdot R_{F_8^{(5)}}(a,b,c,d,e)), \tag{121}
$$

where

$$
F_8 = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & w & w^2 & w^3 & w^4 & w^5 & w^6 & w^7 \\
1 & w^2 & w^4 & w^6 & 1 & w^2 & w^4 & w^6 \\
1 & w^3 & w^6 & w & w^4 & w^7 & w^2 & w^5 \\
1 & w^4 & 1 & w^4 & 1 & w^4 & 1 & w^4 \\
1 & w^5 & w^2 & w^7 & w^4 & w & w^6 & w^3 \\
1 & w^6 & w^4 & w^2 & 1 & w^6 & w^4 & w^2 \\
1 & w^7 & w^6 & w^5 & w^4 & w^3 & w^2 & w
\end{bmatrix}, \tag{122}
$$

and $w = \exp(\boldsymbol{i} \cdot 2\pi/8)$, so $w^8 = 1$, $w^4 = -1$, $w^2 = \boldsymbol{i}$, and

$$
R_{F_8^{(5)}}(a,b,c,d,e) = \left[\begin{array}{cccc|cccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & a & b & c & \bullet & a & b & c \\
\bullet & d & \bullet & d & \bullet & d & \bullet & d \\
\bullet & e & b & c-a+e & \bullet & e & b & c-a+e \\ \hline
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & a & b & c & \bullet & a & b & c \\
\bullet & d & \bullet & d & \bullet & d & \bullet & d \\
\bullet & e & b & c-a+e & \bullet & e & b & c-a+e
\end{array}\right]. \tag{123}
$$

The family $F_8^{(5)}$ is self-cognate. The above orbit is permutation equivalent to the orbit constructed with the Diţă's method [22],

$$
\widetilde{F}_8^{(5)}(\alpha_1,\ldots,\alpha_5) = \left[\begin{array}{c|c}
[F_2]_{1,1} \cdot F_4^{(1)}(\alpha_1) & [F_2]_{1,2} \cdot D(\alpha_3,\alpha_4,\alpha_5) \cdot F_4^{(1)}(\alpha_2) \\ \hline
[F_2]_{2,1} \cdot F_4^{(1)}(\alpha_1) & [F_2]_{2,2} \cdot D(\alpha_3,\alpha_4,\alpha_5) \cdot F_4^{(1)}(\alpha_2)
\end{array}\right], \tag{124}
$$

where

$$F_4^{(1)}(\alpha) \;=\; \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \boldsymbol{i}e^{\boldsymbol{i}\alpha} & -1 & -\boldsymbol{i}e^{\boldsymbol{i}\alpha} \\ 1 & -1 & 1 & -1 \\ 1 & -\boldsymbol{i}e^{\boldsymbol{i}\alpha} & -1 & \boldsymbol{i}e^{\boldsymbol{i}\alpha} \end{bmatrix} \tag{125}$$

is the only maximal affine Hadamard family stemming from F_4 and $D(\alpha_3, \alpha_4, \alpha_5)$ is the 4×4 diagonal matrix $\mathrm{diag}(1, e^{\boldsymbol{i}\alpha_3}, e^{\boldsymbol{i}\alpha_4}, e^{\boldsymbol{i}\alpha_5})$.

Eq. (124) leads to $F_2 \otimes F_4$ for $\alpha_1 = \ldots = \alpha_5 = 0$, which is not equivalent to F_8, see [59]. However, $\widetilde{F}_8^{(5)}(0, 0, (1/8)2\pi, (2/8)2\pi, (3/8)2\pi)$ is permutation equivalent to F_8, since

$$F_8 \;=\; \widetilde{F}_8^{(5)}(0, 0, (1/8)2\pi, (2/8)2\pi, (3/8)2\pi) \cdot [e_1, e_3, e_5, e_7, e_2, e_4, e_6, e_8]^T\,, \tag{126}$$

where e_i denotes the i-th column vector of the standard basis of $\mathbb{C}^8$. Thus (124) generates the only maximal affine Hadamard family stemming from permuted F_8.

The matrix $\widetilde{F}_8^{(5)}(\pi/2, \pi/2, 0, 0, 0)$ yields the only real 8×8 Hadamard matrix, up to permutations and multiplying rows and columns by -1. It is dephased, so it is permutation equivalent to $F_2 \otimes F_2 \otimes F_2$.

Therefore all appropriately permuted tensor products of Fourier matrices, $F_2 \otimes F_2 \otimes F_2$, $F_2 \otimes F_4$ and F_8, although inequivalent [59], are connected by the orbit (124).

5.9. $N = 9$

5.9.1. Orbits stemming from F_9

The only maximal affine Hadamard family stemming from F_9 is the 4-parameter orbit,

$$F_9^{(4)}(a, b, c, d) \;=\; F_9 \circ \mathrm{EXP}\Big(\boldsymbol{i} \cdot R_{F_9^{(4)}}(a, b, c, d)\Big)\,, \tag{127}$$

where

$$F_9 \;=\; \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & w & w^2 & w^3 & w^4 & w^5 & w^6 & w^7 & w^8 \\ 1 & w^2 & w^4 & w^6 & w^8 & w & w^3 & w^5 & w^7 \\ 1 & w^3 & w^6 & 1 & w^3 & w^6 & 1 & w^3 & w^6 \\ 1 & w^4 & w^8 & w^3 & w^7 & w^2 & w^6 & w & w^5 \\ 1 & w^5 & w & w^6 & w^2 & w^7 & w^3 & w^8 & w^4 \\ 1 & w^6 & w^3 & 1 & w^6 & w^3 & 1 & w^6 & w^3 \\ 1 & w^7 & w^5 & w^3 & w & w^8 & w^6 & w^4 & w^2 \\ 1 & w^8 & w^7 & w^6 & w^5 & w^4 & w^3 & w^2 & w \end{bmatrix}, \tag{128}$$

with $w = \exp(\boldsymbol{i} \cdot 2\pi/9)$, so $w^9 = 1$, and

$$R_{F_9^{(4)}}(a, b, c, d) \;=\; \left[\begin{array}{ccc|ccc|ccc} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & a & b & \bullet & a & b & \bullet & a & b \\ \bullet & c & d & \bullet & c & d & \bullet & c & d \\ \hline \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & a & b & \bullet & a & b & \bullet & a & b \\ \bullet & c & d & \bullet & c & d & \bullet & c & d \\ \hline \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & a & b & \bullet & a & b & \bullet & a & b \\ \bullet & c & d & \bullet & c & d & \bullet & c & d \end{array}\right]. \tag{129}$$

The orbit $F_9^{(4)}$ is self-cognate. Observe that its dimension is equal to the defect, $d(F_9) = 4$, which follows from (19). It is permutation equivalent to the 4-dimensional orbit passing through a permuted F_9, constructed using the Diţă's method,

$$\widetilde{F}_9^{(4)}(\alpha_1,\ldots,\alpha_4) = \left[\begin{array}{c|c|c} [F_3]_{1,1}\cdot F_3 & [F_3]_{1,2}\cdot D(\alpha_1,\alpha_2)\cdot F_3 & [F_3]_{1,3}\cdot D(\alpha_3,\alpha_4)\cdot F_3 \\ \hline [F_3]_{2,1}\cdot F_3 & [F_3]_{2,2}\cdot D(\alpha_1,\alpha_2)\cdot F_3 & [F_3]_{2,3}\cdot D(\alpha_3,\alpha_4)\cdot F_3 \\ \hline [F_3]_{3,1}\cdot F_3 & [F_3]_{3,2}\cdot D(\alpha_1,\alpha_2)\cdot F_3 & [F_3]_{3,3}\cdot D(\alpha_3,\alpha_4)\cdot F_3 \end{array}\right], \tag{130}$$

where $D(\alpha,\beta)$ is the 3×3 diagonal matrix $\mathrm{diag}(1, e^{\boldsymbol{i}\alpha}, e^{\boldsymbol{i}\beta})$.

The matrix $\widetilde{F}_9^{(4)}(0,0,0,0) = F_3 \otimes F_3$ is not equivalent to F_9 [59], but

$$F_9 \;=\; \widetilde{F}_9^{(4)}\Big((1/9)2\pi, (2/9)2\pi, (2/9)2\pi, (4/9)2\pi\Big)\cdot [e_1, e_4, e_7, e_2, e_5, e_8, e_3, e_6, e_9]^T, \tag{131}$$

where e_i are the standard basis column vectors. Thus both inequivalent permuted matrices, $F_3 \otimes F_3$ and F_9, are connected by the orbit (130).

5.10. $N = 10$

5.10.1. Orbits stemming from F_{10}

The only maximal affine Hadamard families stemming from F_{10} are

$$F_{10}^{(4)}(a,b,c,d) \;=\; F_{10} \circ \mathrm{EXP}\left(\boldsymbol{i}\cdot R_{F_{10}^{(4)}}(a,b,c,d)\right), \tag{132}$$

$$\left(F_{10}^{(4)}(a,b,c,d)\right)^T \;=\; F_{10} \circ \mathrm{EXP}\left(\boldsymbol{i}\cdot \left(R_{F_{10}^{(4)}}(a,b,c,d)\right)^T\right), \tag{133}$$

where

$$F_{10} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & w & w^2 & w^3 & w^4 & w^5 & w^6 & w^7 & w^8 & w^9 \\ 1 & w^2 & w^4 & w^6 & w^8 & 1 & w^2 & w^4 & w^6 & w^8 \\ 1 & w^3 & w^6 & w^9 & w^2 & w^5 & w^8 & w & w^4 & w^7 \\ 1 & w^4 & w^8 & w^2 & w^6 & 1 & w^4 & w^8 & w^2 & w^6 \\ 1 & w^5 & 1 & w^5 & 1 & w^5 & 1 & w^5 & 1 & w^5 \\ 1 & w^6 & w^2 & w^8 & w^4 & 1 & w^6 & w^2 & w^8 & w^4 \\ 1 & w^7 & w^4 & w & w^8 & w^5 & w^2 & w^9 & w^6 & w^3 \\ 1 & w^8 & w^6 & w^4 & w^2 & 1 & w^8 & w^6 & w^4 & w^2 \\ 1 & w^9 & w^8 & w^7 & w^6 & w^5 & w^4 & w^3 & w^2 & w \end{bmatrix}, \tag{134}$$

with $w = \exp(\boldsymbol{i}\cdot 2\pi/10)$, so $w^{10} = 1$, $w^5 = -1$, and

$$R_{F_{10}^{(4)}}(a,b,c,d) = \left[\begin{array}{ccccc|ccccc} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & a & b & c & d & \bullet & a & b & c & d \\ \hline \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & a & b & c & d & \bullet & a & b & c & d \\ \hline \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & a & b & c & d & \bullet & a & b & c & d \\ \hline \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & a & b & c & d & \bullet & a & b & c & d \\ \hline \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & a & b & c & d & \bullet & a & b & c & d \end{array}\right]. \tag{135}$$

The affine Hadamard families $F_{10}^{(4)}$ and $(F_{10}^{(4)})^T$ are cognate. At least one of them must be permutation equivalent to an orbit constructed using the Diţă's method, either:

$$\widetilde{F}_{10A}^{(4)}(\alpha_1,\ldots,\alpha_4) \;=\; \left[\begin{array}{c|c} [F_2]_{1,1}\cdot F_5 & [F_2]_{1,2}\cdot D(\alpha_1,\ldots,\alpha_4)\cdot F_5 \\ \hline [F_2]_{2,1}\cdot F_5 & [F_2]_{2,2}\cdot D(\alpha_1,\ldots,\alpha_4)\cdot F_5 \end{array}\right], \tag{136}$$

where $D(\alpha_1,\ldots,\alpha_4)$ is the 5×5 diagonal matrix $\mathrm{diag}\,(1,e^{\boldsymbol{i}\alpha_1},\ldots,e^{\boldsymbol{i}\alpha_4})$, or

$$\widetilde{F}_{10B}^{(4)}(\beta_1,\ldots,\beta_4) \tag{137}$$

such that its i,j-th 2×2 block is equal to $[F_5]_{i,j}\cdot D(\alpha)\cdot F_2$, where $i,j\in\{1\ldots 5\}$, $D(\alpha)$ is the diagonal matrix $\mathrm{diag}\,(1,e^{\boldsymbol{i}\alpha})$ and $\alpha=0,\beta_1,\ldots,\beta_4$ for $j=1,2,\ldots,5$ respectively.

This is because $\widetilde{F}_{10A}^{(4)}(\mathbf{0})=F_2\otimes F_5$ and $\widetilde{F}_{10B}^{(4)}(\mathbf{0})=F_5\otimes F_2$ are, according to [59], permutation equivalent to F_{10}, so both Diţă's orbits are maximal affine Hadamard families stemming from permuted F_{10}'s.

5.11. $N=11$

5.11.1. Orbits stemming from F_{11}

The Fourier matrix F_{11} is an isolated 11×11 complex Hadamard matrix

$$F_{11} \;=\; F_{11}^{(0)} \;=\; \begin{bmatrix} 1&1&1&1&1&1&1&1&1&1&1\\ 1&w&w^2&w^3&w^4&w^5&w^6&w^7&w^8&w^9&w^{10}\\ 1&w^2&w^4&w^6&w^8&w^{10}&w&w^3&w^5&w^7&w^9\\ 1&w^3&w^6&w^9&w&w^4&w^7&w^{10}&w^2&w^5&w^8\\ 1&w^4&w^8&w&w^5&w^9&w^2&w^6&w^{10}&w^3&w^7\\ 1&w^5&w^{10}&w^4&w^9&w^3&w^8&w^2&w^7&w&w^6\\ 1&w^6&w&w^7&w^2&w^8&w^3&w^9&w^4&w^{10}&w^5\\ 1&w^7&w^3&w^{10}&w^6&w^2&w^9&w^5&w&w^8&w^4\\ 1&w^8&w^5&w^2&w^{10}&w^7&w^4&w&w^9&w^6&w^3\\ 1&w^9&w^7&w^5&w^3&w&w^{10}&w^8&w^6&w^4&w^2\\ 1&w^{10}&w^9&w^8&w^7&w^6&w^5&w^4&w^3&w^2&w \end{bmatrix}, \tag{138}$$

where $w=\exp(\boldsymbol{i}\cdot 2\pi/11)$, so $w^{11}=1$.

5.11.2. Cyclic 11 roots matrices

There are precisely two $\simeq$ equivalence classes of complex Hadamard matrices, inequivalent to F_{11}, associated with the so-called nonclassical 'index 2' cyclic 11 roots. This result is drawn in [21, p. 319].

The classes are represented by the matrices $\widetilde{C}_{11A}^{(0)}$, $\widetilde{C}_{11B}^{(0)}$ below, their respective dephased forms are denoted by $C_{11A}^{(0)}$, $C_{11B}^{(0)}$. Both matrices have the circulant structure $[U]_{i,j}=x_{(i-j \bmod 11)+1}$, where

$$x \;=\; [1,\ 1,\ e,\ 1,\ 1,\ 1,\ e,\ e,\ e,\ 1,\ e] \quad \text{for} \quad \widetilde{C}_{11A}^{(0)} \tag{139}$$

$$x \;=\; [1,\ 1,\ \overline{e},\ 1,\ 1,\ 1,\ \overline{e},\ \overline{e},\ \overline{e},\ 1,\ \overline{e}] \quad \text{for} \quad \widetilde{C}_{11B}^{(0)} \tag{140}$$

and

$$e \;=\; -\frac{5}{6}+\boldsymbol{i}\cdot\frac{\sqrt{11}}{6}\,. \tag{141}$$

There holds

$$\widetilde{C}_{11B}^{(0)} = \overline{\widetilde{C}_{11A}^{(0)}} \quad \text{so} \quad C_{11B}^{(0)} = \overline{C_{11A}^{(0)}}. \tag{142}$$

Applying transposition to $\widetilde{C}_{11A}^{(0)}$, $\widetilde{C}_{11B}^{(0)}$ — equivalently to $C_{11A}^{(0)}$, $C_{11B}^{(0)}$ — yields a matrix equivalent to the original one (see (47)). The matrices are given by:

$$\widetilde{C}_{11A}^{(0)} = \begin{bmatrix} 1 & e & 1 & e & e & e & 1 & 1 & 1 & e & 1 \\ 1 & 1 & e & 1 & e & e & e & 1 & 1 & 1 & e \\ e & 1 & 1 & e & 1 & e & e & e & 1 & 1 & 1 \\ 1 & e & 1 & 1 & e & 1 & e & e & e & 1 & 1 \\ 1 & 1 & e & 1 & 1 & e & 1 & e & e & e & 1 \\ 1 & 1 & 1 & e & 1 & 1 & e & 1 & e & e & e \\ e & 1 & 1 & 1 & e & 1 & 1 & e & 1 & e & e \\ e & e & 1 & 1 & 1 & e & 1 & 1 & e & 1 & e \\ e & e & e & 1 & 1 & 1 & e & 1 & 1 & e & 1 \\ 1 & e & e & e & 1 & 1 & 1 & e & 1 & 1 & e \\ e & 1 & e & e & e & 1 & 1 & 1 & e & 1 & 1 \end{bmatrix}, \tag{143}$$

$$\widetilde{C}_{11B}^{(0)} = \begin{bmatrix} 1 & e^{-1} & 1 & e^{-1} & e^{-1} & e^{-1} & 1 & 1 & 1 & e^{-1} & 1 \\ 1 & 1 & e^{-1} & 1 & e^{-1} & e^{-1} & e^{-1} & 1 & 1 & 1 & e^{-1} \\ e^{-1} & 1 & 1 & e^{-1} & 1 & e^{-1} & e^{-1} & e^{-1} & 1 & 1 & 1 \\ 1 & e^{-1} & 1 & 1 & e^{-1} & 1 & e^{-1} & e^{-1} & e^{-1} & 1 & 1 \\ 1 & 1 & e^{-1} & 1 & 1 & e^{-1} & 1 & e^{-1} & e^{-1} & e^{-1} & 1 \\ 1 & 1 & 1 & e^{-1} & 1 & 1 & e^{-1} & 1 & e^{-1} & e^{-1} & e^{-1} \\ e^{-1} & 1 & 1 & 1 & e^{-1} & 1 & 1 & e^{-1} & 1 & e^{-1} & e^{-1} \\ e^{-1} & e^{-1} & 1 & 1 & 1 & e^{-1} & 1 & 1 & e^{-1} & 1 & e^{-1} \\ e^{-1} & e^{-1} & e^{-1} & 1 & 1 & 1 & e^{-1} & 1 & 1 & e^{-1} & 1 \\ 1 & e^{-1} & e^{-1} & e^{-1} & 1 & 1 & 1 & e^{-1} & 1 & 1 & e^{-1} \\ e^{-1} & 1 & e^{-1} & e^{-1} & e^{-1} & 1 & 1 & 1 & e^{-1} & 1 & 1 \end{bmatrix}, \tag{144}$$

$$C_{11A}^{(0)} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & e^{-1} & e & e^{-1} & 1 & 1 & e & 1 & 1 & e^{-1} & e \\ 1 & e^{-2} & e^{-1} & e^{-1} & e^{-2} & e^{-1} & 1 & 1 & e^{-1} & e^{-2} & e^{-1} \\ 1 & 1 & 1 & e^{-1} & 1 & e^{-1} & e & e & e & e^{-1} & 1 \\ 1 & e^{-1} & e & e^{-1} & e^{-1} & 1 & 1 & e & e & 1 & 1 \\ 1 & e^{-1} & 1 & 1 & e^{-1} & e^{-1} & e & 1 & e & 1 & e \\ 1 & e^{-2} & e^{-1} & e^{-2} & e^{-1} & e^{-2} & e^{-1} & 1 & e^{-1} & e^{-1} & 1 \\ 1 & e^{-1} & e^{-1} & e^{-2} & e^{-2} & e^{-1} & e^{-1} & e^{-1} & 1 & e^{-2} & 1 \\ 1 & e^{-1} & 1 & e^{-2} & e^{-2} & e^{-2} & 1 & e^{-1} & e^{-1} & e^{-1} & e^{-1} \\ 1 & 1 & e & 1 & e^{-1} & e^{-1} & 1 & e & 1 & e^{-1} & e \\ 1 & e^{-2} & 1 & e^{-1} & e^{-1} & e^{-2} & e^{-1} & e^{-1} & 1 & e^{-2} & e^{-1} \end{bmatrix}, \tag{145}$$

$$C_{11B}^{(0)} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & e & e^{-1} & e & 1 & 1 & e^{-1} & 1 & 1 & e & e^{-1} \\ 1 & e^{2} & e & e & e^{2} & e & 1 & 1 & e & e^{2} & e \\ 1 & 1 & 1 & e & 1 & e & e^{-1} & e^{-1} & e^{-1} & e & 1 \\ 1 & e & e^{-1} & e & e & 1 & 1 & e^{-1} & e^{-1} & 1 & 1 \\ 1 & e & 1 & 1 & e & e & e^{-1} & 1 & e^{-1} & 1 & e^{-1} \\ 1 & e^{2} & e & e^{2} & e & e^{2} & e & 1 & e & e & 1 \\ 1 & e & e & e^{2} & e^{2} & e & e & e & 1 & e^{2} & 1 \\ 1 & e & 1 & e^{2} & e^{2} & e^{2} & 1 & e & e & e & e \\ 1 & 1 & e^{-1} & 1 & e & e & 1 & e^{-1} & 1 & e & e^{-1} \\ 1 & e^{2} & 1 & e & e & e^{2} & e & e & 1 & e^{2} & e \end{bmatrix}. \tag{146}$$

5.11.3. Nicoara's 11×11 *complex Hadamard matrix*

Another equivalence class of 11×11 Hadamard matrices is represented by the matrix communicated to the authors by Nicoara:

$$N_{11}^{(0)} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & a & -a & -a & -a & -1 & -1 & -a^{-1} & -a^{-1} & -1 & -1 \\ 1 & -a & a & -a & -a & -1 & -1 & -1 & -1 & -a^{-1} & -a^{-1} \\ 1 & -a & -a & -1 & a & -1 & -a & -a^{-1} & -1 & -a^{-1} & -1 \\ 1 & -a & -a & a & -1 & -a & -1 & -1 & -a^{-1} & -1 & -a^{-1} \\ 1 & -1 & -1 & -1 & -a & -a & 1 & -a^{-1} & -1 & -1 & -a^{-1} \\ 1 & -1 & -1 & -a & -1 & 1 & -a & -1 & -a^{-1} & -a^{-1} & -1 \\ 1 & -a^{-1} & -1 & -a^{-1} & -1 & -a^{-1} & -1 & -a^{-1} & a^{-1} & -a^{-2} & -a^{-2} \\ 1 & -a^{-1} & -1 & -1 & -a^{-1} & -1 & -a^{-1} & a^{-1} & -a^{-1} & -a^{-2} & -a^{-2} \\ 1 & -1 & -a^{-1} & -a^{-1} & -1 & -1 & -a^{-1} & -a^{-2} & -a^{-2} & -a^{-2} & a^{-2} \\ 1 & -1 & -a^{-1} & -1 & -a^{-1} & -a^{-1} & -1 & -a^{-2} & -a^{-2} & a^{-2} & -a^{-2} \end{bmatrix}, \quad (147)$$

where

$$a = -\frac{3}{4} - \boldsymbol{i} \cdot \frac{\sqrt{7}}{4}\,. \qquad (148)$$

The above matrix is isolated[7] since its defect $d(N_{11}^{(0)}) = 0$.

5.12. $N = 12$

5.12.1. Orbits stemming from F_{12}

The only maximal affine Hadamard families stemming from F_{12} are:

$$F_{12A}^{(9)}(a,b,c,d,e,f,g,h,i) = F_{12} \circ \mathrm{EXP}\left(\boldsymbol{i} \cdot R_{F_{12A}^{(9)}}(a,\ldots,i)\right), \qquad (149)$$

$$F_{12B}^{(9)}(a,b,c,d,e,f,g,h,i) = F_{12} \circ \mathrm{EXP}\left(\boldsymbol{i} \cdot R_{F_{12B}^{(9)}}(a,\ldots,i)\right), \qquad (150)$$

$$F_{12C}^{(9)}(a,b,c,d,e,f,g,h,i) = F_{12} \circ \mathrm{EXP}\left(\boldsymbol{i} \cdot R_{F_{12C}^{(9)}}(a,\ldots,i)\right), \qquad (151)$$

$$F_{12D}^{(9)}(a,b,c,d,e,f,g,h,i) = F_{12} \circ \mathrm{EXP}\left(\boldsymbol{i} \cdot R_{F_{12D}^{(9)}}(a,\ldots,i)\right), \qquad (152)$$

$$\left(F_{12B}^{(9)}(a,b,c,d,e,f,g,h,i)\right)^T = F_{12} \circ \mathrm{EXP}\left(\boldsymbol{i} \cdot \left(R_{F_{12B}^{(9)}}(a,\ldots,i)\right)^T\right),$$

$$\left(F_{12C}^{(9)}(a,b,c,d,e,f,g,h,i)\right)^T = F_{12} \circ \mathrm{EXP}\left(\boldsymbol{i} \cdot \left(R_{F_{12C}^{(9)}}(a,\ldots,i)\right)^T\right),$$

$$\left(F_{12D}^{(9)}(a,b,c,d,e,f,g,h,i)\right)^T = F_{12} \circ \mathrm{EXP}\left(\boldsymbol{i} \cdot \left(R_{F_{12D}^{(9)}}(a,\ldots,i)\right)^T\right),$$

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[7]R. Nicoara, private communication

where

$$F_{12} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & w & w^2 & w^3 & w^4 & w^5 & w^6 & w^7 & w^8 & w^9 & w^{10} & w^{11} \\ 1 & w^2 & w^4 & w^6 & w^8 & w^{10} & 1 & w^2 & w^4 & w^6 & w^8 & w^{10} \\ 1 & w^3 & w^6 & w^9 & 1 & w^3 & w^6 & w^9 & 1 & w^3 & w^6 & w^9 \\ 1 & w^4 & w^8 & 1 & w^4 & w^8 & 1 & w^4 & w^8 & 1 & w^4 & w^8 \\ 1 & w^5 & w^{10} & w^3 & w^8 & w & w^6 & w^{11} & w^4 & w^9 & w^2 & w^7 \\ 1 & w^6 & 1 & w^6 & 1 & w^6 & 1 & w^6 & 1 & w^6 & 1 & w^6 \\ 1 & w^7 & w^2 & w^9 & w^4 & w^{11} & w^6 & w & w^8 & w^3 & w^{10} & w^5 \\ 1 & w^8 & w^4 & 1 & w^8 & w^4 & 1 & w^8 & w^4 & 1 & w^8 & w^4 \\ 1 & w^9 & w^6 & w^3 & 1 & w^9 & w^6 & w^3 & 1 & w^9 & w^6 & w^3 \\ 1 & w^{10} & w^8 & w^6 & w^4 & w^2 & 1 & w^{10} & w^8 & w^6 & w^4 & w^2 \\ 1 & w^{11} & w^{10} & w^9 & w^8 & w^7 & w^6 & w^5 & w^4 & w^3 & w^2 & w \end{bmatrix} \tag{153}$$

and $w = \exp(\boldsymbol{i} \cdot 2\pi/12)$, so $w^{12} = 1$, $w^6 = -1$, $w^3 = \boldsymbol{i}$, and

$$R_{F_{12A}^{(9)}}(a, \ldots, i) = \left[\begin{array}{cccccc|cccccc} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & a & b & c & d & e & \bullet & a & b & c & d & e \\ \bullet & f & \bullet & f & \bullet & f & \bullet & f & \bullet & f & \bullet & f \\ \bullet & g & b & c-a+g & d & e-a+g & \bullet & g & b & c-a+g & d & e-a+g \\ \bullet & h & \bullet & h & \bullet & h & \bullet & h & \bullet & h & \bullet & h \\ \bullet & i & b & c-a+i & d & e-a+i & \bullet & i & b & c-a+i & d & e-a+i \\ \hline \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & a & b & c & d & e & \bullet & a & b & c & d & e \\ \bullet & f & \bullet & f & \bullet & f & \bullet & f & \bullet & f & \bullet & f \\ \bullet & g & b & c-a+g & d & e-a+g & \bullet & g & b & c-a+g & d & e-a+g \\ \bullet & h & \bullet & h & \bullet & h & \bullet & h & \bullet & h & \bullet & h \\ \bullet & i & b & c-a+i & d & e-a+i & \bullet & i & b & c-a+i & d & e-a+i \end{array}\right], \tag{154}$$

$$R_{F_{12B}^{(9)}}(a, \ldots, i) = \left[\begin{array}{cccccc|cccccc} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & a & b & c & d & e & \bullet & a & b & c & d & e \\ \bullet & f & g & \bullet & f & g & \bullet & f & g & \bullet & f & g \\ \bullet & h & i & c & d-a+h & e-b+i & \bullet & h & i & c & d-a+h & e-b+i \\ \hline \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & a & b & c & d & e & \bullet & a & b & c & d & e \\ \bullet & f & g & \bullet & f & g & \bullet & f & g & \bullet & f & g \\ \bullet & h & i & c & d-a+h & e-b+i & \bullet & h & i & c & d-a+h & e-b+i \\ \hline \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & a & b & c & d & e & \bullet & a & b & c & d & e \\ \bullet & f & g & \bullet & f & g & \bullet & f & g & \bullet & f & g \\ \bullet & h & i & c & d-a+h & e-b+i & \bullet & h & i & c & d-a+h & e-b+i \end{array}\right], \tag{155}$$

$$R_{F_{12C}^{(9)}}(a, \ldots, i) = \left[\begin{array}{cccccccccccc} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & a & b & c & \bullet & d & b & a & \bullet & c & b & d \\ \bullet & e & f & e & \bullet & e & f & e & \bullet & e & f & e \\ \bullet & g & \bullet & c-a+g & \bullet & d-a+g & \bullet & g & \bullet & c-a+g & \bullet & d-a+g \\ \bullet & h & b & h & \bullet & h & b & h & \bullet & h & b & h \\ \bullet & i & f & c-a+i & \bullet & d-a+i & f & i & \bullet & c-a+i & f & d-a+i \\ \hline \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & a & b & c & \bullet & d & b & a & \bullet & c & b & d \\ \bullet & e & f & e & \bullet & e & f & e & \bullet & e & f & e \\ \bullet & g & \bullet & c-a+g & \bullet & d-a+g & \bullet & g & \bullet & c-a+g & \bullet & d-a+g \\ \bullet & h & b & h & \bullet & h & b & h & \bullet & h & b & h \\ \bullet & i & f & c-a+i & \bullet & d-a+i & f & i & \bullet & c-a+i & f & d-a+i \end{array}\right], \tag{156}$$

$$R_{F_{12D}^{(9)}}(a,\ldots,i)=\left[\begin{array}{cccccccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & a & b & c & d & a & \bullet & c & b & a & d & c \\
\bullet & e & \bullet & f & \bullet & e & \bullet & f & \bullet & e & \bullet & f \\
\bullet & g & b & g & d & g & \bullet & g & b & g & d & g \\
\bullet & h & \bullet & c-a+h & \bullet & h & \bullet & c-a+h & \bullet & h & \bullet & c-a+h \\
\bullet & i & b & f-e+i & d & i & \bullet & f-e+i & b & i & d & f-e+i \\
\hline
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & a & b & c & d & a & \bullet & c & b & a & d & c \\
\bullet & e & \bullet & f & \bullet & e & \bullet & f & \bullet & e & \bullet & f \\
\bullet & g & b & g & d & g & \bullet & g & b & g & d & g \\
\bullet & h & \bullet & c-a+h & \bullet & h & \bullet & c-a+h & \bullet & h & \bullet & c-a+h \\
\bullet & i & b & f-e+i & d & i & \bullet & f-e+i & b & i & d & f-e+i
\end{array}\right]. \tag{157}$$

Thus we have three pairs of cognate families, and the $F_{12A}^{(9)}$ family is self-cognate. At least one of the above orbits is permutation equivalent to the orbit constructed using the Diţă's method,

$$\widetilde{F}_{12}^{(9)}(\alpha_1,\ldots,\alpha_9) = \tag{158}$$
$$\left[\begin{array}{c|c|c}
[F_3]_{1,1}\cdot F_4^{(1)}(\alpha_1) & [F_3]_{1,2}\cdot D(\alpha_4,\alpha_5,\alpha_6)\cdot F_4^{(1)}(\alpha_2) & [F_3]_{1,3}\cdot D(\alpha_7,\alpha_8,\alpha_9)\cdot F_4^{(1)}(\alpha_3) \\ \hline
[F_3]_{2,1}\cdot F_4^{(1)}(\alpha_1) & [F_3]_{2,2}\cdot D(\alpha_4,\alpha_5,\alpha_6)\cdot F_4^{(1)}(\alpha_2) & [F_3]_{2,3}\cdot D(\alpha_7,\alpha_8,\alpha_9)\cdot F_4^{(1)}(\alpha_3) \\ \hline
[F_3]_{3,1}\cdot F_4^{(1)}(\alpha_1) & [F_3]_{3,2}\cdot D(\alpha_4,\alpha_5,\alpha_6)\cdot F_4^{(1)}(\alpha_2) & [F_3]_{3,3}\cdot D(\alpha_7,\alpha_8,\alpha_9)\cdot F_4^{(1)}(\alpha_3)
\end{array}\right],$$

where $D(\alpha,\beta,\gamma)$ denotes the 4×4 diagonal matrix $\mathrm{diag}\,(1,e^{\boldsymbol{i}\alpha},e^{\boldsymbol{i}\beta},e^{\boldsymbol{i}\gamma})$ and $F_4^{(1)}(\alpha)$ is given by (62).

It passes through $\widetilde{F}_{12}^{(9)}(\mathbf{0}) = F_3\otimes F_4$, which is, according to [59], permutation equivalent to F_{12}, so $\widetilde{F}_{12}^{(9)}$ is a maximal affine Hadamard family stemming from a permuted F_{12}. It also passes through $\widetilde{F}_{12}^{(9)}(\pi/2,\pi/2,\pi/2,\mathbf{0})$, which is dephased, so it is a permuted $F_3\otimes F_2\otimes F_2$. Thus permuted and inequivalent F_{12} and $F_3\otimes F_2\otimes F_2$ (see [59]) are connected by the orbit of (158).

Note also that a similar construction using the Diţă's method with the role of F_3 and $F_4^{(1)}$ exchanged yields a 7-dimensional orbit, also passing through a permuted F_{12}: $F_4\otimes F_3$, which is a suborbit of one of the existing 9-dimensional maximal affine Hadamard families stemming from $F_4\otimes F_3$.

5.12.2. *Other* 12×12 *orbits*

Other dephased 12×12 orbits can be obtained, using the Diţă's method, from F_2 and dephased 6×6 complex Hadamard matrices from section 5.6., for example:

$$FD_{12}^{(8)}(\alpha_1,\ldots,\alpha_8) = \left[\begin{array}{c|c}
[F_2]_{1,1}\cdot F_6^{(2)}(\alpha_1,\alpha_2) & [F_2]_{1,2}\cdot D(\alpha_4,\ldots,\alpha_8)\cdot D_6^{(1)}(\alpha_3) \\ \hline
[F_2]_{2,1}\cdot F_6^{(2)}(\alpha_1,\alpha_2) & [F_2]_{2,2}\cdot D(\alpha_4,\ldots,\alpha_8)\cdot D_6^{(1)}(\alpha_3)
\end{array}\right],$$

$$FC_{12}^{(7)}(\alpha_1,\ldots,\alpha_7) = \left[\begin{array}{c|c}
[F_2]_{1,1}\cdot F_6^{(2)}(\alpha_1,\alpha_2) & [F_2]_{1,2}\cdot D(\alpha_3,\ldots,\alpha_7)\cdot C_6^{(0)} \\ \hline
[F_2]_{2,1}\cdot F_6^{(2)}(\alpha_1,\alpha_2) & [F_2]_{2,2}\cdot D(\alpha_3,\ldots,\alpha_7)\cdot C_6^{(0)}
\end{array}\right],$$

$$FS_{12}^{(7)}(\alpha_1,\ldots,\alpha_7) = \left[\begin{array}{c|c}
[F_2]_{1,1}\cdot F_6^{(2)}(\alpha_1,\alpha_2) & [F_2]_{1,2}\cdot D(\alpha_3,\ldots,\alpha_7)\cdot S_6^{(0)} \\ \hline
[F_2]_{2,1}\cdot F_6^{(2)}(\alpha_1,\alpha_2) & [F_2]_{2,2}\cdot D(\alpha_3,\ldots,\alpha_7)\cdot S_6^{(0)}
\end{array}\right],$$

$$DD_{12}^{(7)}(\alpha_1,\ldots,\alpha_7) = \left[\begin{array}{c|c}
[F_2]_{1,1}\cdot D_6^{(1)}(\alpha_1) & [F_2]_{1,2}\cdot D(\alpha_3,\ldots,\alpha_7)\cdot D_6^{(1)}(\alpha_2) \\ \hline
[F_2]_{2,1}\cdot D_6^{(1)}(\alpha_1) & [F_2]_{2,2}\cdot D(\alpha_3,\ldots,\alpha_7)\cdot D_6^{(1)}(\alpha_2)
\end{array}\right],$$

$$DC_{12}^{(6)}(\alpha_1,\ldots,\alpha_6) = \left[\begin{array}{c|c}
[F_2]_{1,1}\cdot D_6^{(1)}(\alpha_1) & [F_2]_{1,2}\cdot D(\alpha_2,\ldots,\alpha_6)\cdot C_6^{(0)} \\ \hline
[F_2]_{2,1}\cdot D_6^{(1)}(\alpha_1) & [F_2]_{2,2}\cdot D(\alpha_2,\ldots,\alpha_6)\cdot C_6^{(0)}
\end{array}\right],$$

$$
\begin{aligned}
DS_{12}^{(6)}(\alpha_1,\dots,\alpha_6) &= \left[\begin{array}{c|c} [F_2]_{1,1}\cdot D_6^{(1)}(\alpha_1) & [F_2]_{1,2}\cdot D(\alpha_2,\dots,\alpha_6)\cdot S_6^{(0)} \\ \hline [F_2]_{2,1}\cdot D_6^{(1)}(\alpha_1) & [F_2]_{2,2}\cdot D(\alpha_2,\dots,\alpha_6)\cdot S_6^{(0)} \end{array}\right], \\
CC_{12}^{(5)}(\alpha_1,\dots,\alpha_5) &= \left[\begin{array}{c|c} [F_2]_{1,1}\cdot C_6^{(0)} & [F_2]_{1,2}\cdot D(\alpha_1,\dots,\alpha_5)\cdot C_6^{(0)} \\ \hline [F_2]_{2,1}\cdot C_6^{(0)} & [F_2]_{2,2}\cdot D(\alpha_1,\dots,\alpha_5)\cdot C_6^{(0)} \end{array}\right], \\
CS_{12}^{(5)}(\alpha_1,\dots,\alpha_5) &= \left[\begin{array}{c|c} [F_2]_{1,1}\cdot C_6^{(0)} & [F_2]_{1,2}\cdot D(\alpha_1,\dots,\alpha_5)\cdot S_6^{(0)} \\ \hline [F_2]_{2,1}\cdot C_6^{(0)} & [F_2]_{2,2}\cdot D(\alpha_1,\dots,\alpha_5)\cdot S_6^{(0)} \end{array}\right], \\
SS_{12}^{(5)}(\alpha_1,\dots,\alpha_5) &= \left[\begin{array}{c|c} [F_2]_{1,1}\cdot S_6^{(0)} & [F_2]_{1,2}\cdot D(\alpha_1,\dots,\alpha_5)\cdot S_6^{(0)} \\ \hline [F_2]_{2,1}\cdot S_6^{(0)} & [F_2]_{2,2}\cdot D(\alpha_1,\dots,\alpha_5)\cdot S_6^{(0)} \end{array}\right],
\end{aligned}
$$

where $D(\beta_1,\dots,\beta_5)$ denotes the 6×6 diagonal matrix $\mathrm{diag}\,(1, e^{\boldsymbol{i}\beta_1},\dots,e^{\boldsymbol{i}\beta_5})$.

5.13. $N = 13$

5.13.1. Orbits stemming from F_{13}

The Fourier matrix F_{13} is an isolated 13×13 complex Hadamard matrix

$$
F_{13} = F_{13}^{(0)} = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & w & w^2 & w^3 & w^4 & w^5 & w^6 & w^7 & w^8 & w^9 & w^{10} & w^{11} & w^{12} \\
1 & w^2 & w^4 & w^6 & w^8 & w^{10} & w^{12} & w & w^3 & w^5 & w^7 & w^9 & w^{11} \\
1 & w^3 & w^6 & w^9 & w^{12} & w^2 & w^5 & w^8 & w^{11} & w & w^4 & w^7 & w^{10} \\
1 & w^4 & w^8 & w^{12} & w^3 & w^7 & w^{11} & w^2 & w^6 & w^{10} & w & w^5 & w^9 \\
1 & w^5 & w^{10} & w^2 & w^7 & w^{12} & w^4 & w^9 & w & w^6 & w^{11} & w^3 & w^8 \\
1 & w^6 & w^{12} & w^5 & w^{11} & w^4 & w^{10} & w^3 & w^9 & w^2 & w^8 & w & w^7 \\
1 & w^7 & w & w^8 & w^2 & w^9 & w^3 & w^{10} & w^4 & w^{11} & w^5 & w^{12} & w^6 \\
1 & w^8 & w^3 & w^{11} & w^6 & w & w^9 & w^4 & w^{12} & w^7 & w^2 & w^{10} & w^5 \\
1 & w^9 & w^5 & w & w^{10} & w^6 & w^2 & w^{11} & w^7 & w^3 & w^{12} & w^8 & w^4 \\
1 & w^{10} & w^7 & w^4 & w & w^{11} & w^8 & w^5 & w^2 & w^{12} & w^9 & w^6 & w^3 \\
1 & w^{11} & w^9 & w^7 & w^5 & w^3 & w & w^{12} & w^{10} & w^8 & w^6 & w^4 & w^2 \\
1 & w^{12} & w^{11} & w^{10} & w^9 & w^8 & w^7 & w^6 & w^5 & w^4 & w^3 & w^2 & w
\end{bmatrix}, \tag{159}
$$

where $w = \exp(\boldsymbol{i}\cdot 2\pi/13)$, so $w^{13} = 1$.

5.13.2. Petrescu 13×13 *orbit*

There exists a continuous 2-parameter orbit $P_{13}^{(2)}$ of 13×13 complex Hadamard matrices found by Petrescu [62],

$$
P_{13}^{(2)}(e,f) = P_{13}\circ \mathrm{EXP}\left(\boldsymbol{i}\cdot R_{P_{13}^{(2)}}(e,f)\right), \tag{160}
$$

where

$$
P_{13} = \left[\begin{array}{c|cccc|cccc|cccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline
1 & -1 & t^{10} & -t^5 & t^5 & \boldsymbol{i}t^5 & -\boldsymbol{i}t^5 & \boldsymbol{i}t^{15} & -\boldsymbol{i}t^{15} & t^{16} & t^4 & t^{22} & t^{28} \\
1 & t^{10} & -1 & t^5 & -t^5 & -\boldsymbol{i}t^5 & \boldsymbol{i}t^5 & -\boldsymbol{i}t^{15} & \boldsymbol{i}t^{15} & t^{16} & t^4 & t^{22} & t^{28} \\
1 & -t^5 & t^5 & -1 & t^{10} & \boldsymbol{i}t^{15} & -\boldsymbol{i}t^{15} & \boldsymbol{i}t^5 & -\boldsymbol{i}t^5 & t^4 & t^{16} & t^{28} & t^{22} \\
1 & t^5 & -t^5 & t^{10} & -1 & -\boldsymbol{i}t^{15} & \boldsymbol{i}t^{15} & -\boldsymbol{i}t^5 & \boldsymbol{i}t^5 & t^4 & t^{16} & t^{28} & t^{22} \\ \hline
1 & \boldsymbol{i}t^5 & -\boldsymbol{i}t^5 & \boldsymbol{i}t^{25} & -\boldsymbol{i}t^{25} & -1 & t^{10} & -t^5 & t^5 & t^{22} & t^{28} & t^4 & t^{16} \\
1 & -\boldsymbol{i}t^5 & \boldsymbol{i}t^5 & -\boldsymbol{i}t^{25} & \boldsymbol{i}t^{25} & t^{10} & -1 & t^5 & -t^5 & t^{22} & t^{28} & t^4 & t^{16} \\
1 & \boldsymbol{i}t^{25} & -\boldsymbol{i}t^{25} & \boldsymbol{i}t^5 & -\boldsymbol{i}t^5 & -t^5 & t^5 & -1 & t^{10} & t^{28} & t^{22} & t^{16} & t^4 \\
1 & -\boldsymbol{i}t^{25} & \boldsymbol{i}t^{25} & -\boldsymbol{i}t^5 & \boldsymbol{i}t^5 & t^5 & -t^5 & t^{10} & -1 & t^{28} & t^{22} & t^{16} & t^4 \\ \hline
1 & t^4 & t^4 & t^{16} & t^{16} & t^{28} & t^{28} & t^{22} & t^{22} & t^{20} & t^{10} & t^{10} & t^{10} \\
1 & t^{16} & t^{16} & t^4 & t^4 & t^{22} & t^{22} & t^{28} & t^{28} & t^{10} & t^{20} & t^{10} & t^{10} \\
1 & t^{28} & t^{28} & t^{22} & t^{22} & t^{16} & t^{16} & t^4 & t^4 & t^{10} & t^{10} & t^{20} & t^{10} \\
1 & t^{22} & t^{22} & t^{28} & t^{28} & t^4 & t^4 & t^{16} & t^{16} & t^{10} & t^{10} & t^{10} & t^{20}
\end{array}\right], \tag{161}
$$

and $t = \exp(\boldsymbol{i} \cdot 2\pi/30)$, so $t^{30} = 1,\ t^{15} = -1$, and

$$R_{P_{13}^{(2)}}(e,f) \ = \tag{162}$$

$$\left[\begin{array}{c|cccc|cccc|cccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\hline
\bullet & \bullet & \bullet & f & f & e & e & e+G(f) & e+G(f) & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & f & f & e & e & e+G(f) & e+G(f) & \bullet & \bullet & \bullet & \bullet \\
\bullet & f & f & \bullet & \bullet & e+G(f) & e+G(f) & e & e & \bullet & \bullet & \bullet & \bullet \\
\bullet & f & f & \bullet & \bullet & e+G(f) & e+G(f) & e & e & \bullet & \bullet & \bullet & \bullet \\
\hline
\bullet & -e & -e & -e-G(f) & -e-G(f) & \bullet & \bullet & -f & -f & \bullet & \bullet & \bullet & \bullet \\
\bullet & -e & -e & -e-G(f) & -e-G(f) & \bullet & \bullet & -f & -f & \bullet & \bullet & \bullet & \bullet \\
\bullet & -e-G(f) & -e-G(f) & -e & -e & -f & -f & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & -e-G(f) & -e-G(f) & -e & -e & -f & -f & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\hline
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet
\end{array}\right],$$

where

$$G(f) \ = \ \arg\Big(-\frac{\cos(f)}{2} + \boldsymbol{i} \cdot \frac{\sqrt{2}}{4}\sqrt{7-\cos(2f)}\Big) - \frac{2\pi}{3}\,. \tag{163}$$

Since the function $G(f)$ is nonlinear, the above family is not an affine Hadamard family, but it is not clear whether it could be contained in any affine Hadamard family of a larger dimension.

Due to some freedom in the construction of family components the method of Petrescu allows one to build other similar families of Hadamard matrices. Not knowing whether they are inequivalent we are not going to consider them here.

5.13.3. Cyclic 13 roots matrices

There are precisely two $\simeq$ equivalence classes of complex Hadamard matrices, inequivalent to F_{13}, associated with the so-called 'index 2' cyclic 13 roots. This result is drawn in [21, p. 319].

The classes are represented by the matrices $\widetilde{C}_{13A}^{(0)}$, $\widetilde{C}_{13B}^{(0)}$ below, their respective dephased forms are denoted by $C_{13A}^{(0)}$, $C_{13B}^{(0)}$. Both matrices have the circulant structure $[U]_{i,j} = x_{(i-j \bmod 13)+1}$, where

$$x \ = \ [1,\ c,\ \overline{c},\ c,\ c,\ \overline{c},\ \overline{c},\ \overline{c},\ \overline{c},\ c,\ c,\ \overline{c},\ c] \quad \text{for} \quad \widetilde{C}_{13A}^{(0)} \tag{164}$$

$$x \ = \ [1,\ d,\ \overline{d},\ d,\ d,\ \overline{d},\ \overline{d},\ \overline{d},\ \overline{d},\ d,\ d,\ \overline{d},\ d] \quad \text{for} \quad \widetilde{C}_{13B}^{(0)} \tag{165}$$

and

$$c \ = \ \Big(\frac{-1+\sqrt{13}}{12}\Big) + \boldsymbol{i} \cdot \Big(\frac{\sqrt{130+2\sqrt{13}}}{12}\Big)\,, \tag{166}$$

$$d \ = \ \Big(\frac{-1-\sqrt{13}}{12}\Big) + \boldsymbol{i} \cdot \Big(\frac{\sqrt{130-2\sqrt{13}}}{12}\Big)\,. \tag{167}$$

Conjugating the $C_{13k}^{(0)},\ \widetilde{C}_{13k}^{(0)},\ k = A, B$ yields a matrix equivalent to the original one [21]. The matrices $C_{13A}^{(0)}$, $\widetilde{C}_{13A}^{(0)}$ and $C_{13B}^{(0)}$, $\widetilde{C}_{13B}^{(0)}$ are symmetric. They read

$$\widetilde{C}_{13A}^{(0)} = \begin{bmatrix} 1 & c & c^{-1} & c & c & c^{-1} & c^{-1} & c^{-1} & c^{-1} & c & c & c^{-1} & c \\ c & 1 & c & c^{-1} & c & c & c^{-1} & c^{-1} & c^{-1} & c^{-1} & c & c & c^{-1} \\ c^{-1} & c & 1 & c & c^{-1} & c & c & c^{-1} & c^{-1} & c^{-1} & c^{-1} & c & c \\ c & c^{-1} & c & 1 & c & c^{-1} & c & c & c^{-1} & c^{-1} & c^{-1} & c^{-1} & c \\ c & c & c^{-1} & c & 1 & c & c^{-1} & c & c & c^{-1} & c^{-1} & c^{-1} & c^{-1} \\ c^{-1} & c & c & c^{-1} & c & 1 & c & c^{-1} & c & c & c^{-1} & c^{-1} & c^{-1} \\ c^{-1} & c^{-1} & c & c & c^{-1} & c & 1 & c & c^{-1} & c & c & c^{-1} & c^{-1} \\ c^{-1} & c^{-1} & c^{-1} & c & c & c^{-1} & c & 1 & c & c^{-1} & c & c & c^{-1} \\ c^{-1} & c^{-1} & c^{-1} & c^{-1} & c & c & c^{-1} & c & 1 & c & c^{-1} & c & c \\ c & c^{-1} & c^{-1} & c^{-1} & c^{-1} & c & c & c^{-1} & c & 1 & c & c^{-1} & c \\ c & c & c^{-1} & c^{-1} & c^{-1} & c^{-1} & c & c & c^{-1} & c & 1 & c & c^{-1} \\ c^{-1} & c & c & c^{-1} & c^{-1} & c^{-1} & c^{-1} & c & c & c^{-1} & c & 1 & c \\ c & c^{-1} & c & c & c^{-1} & c^{-1} & c^{-1} & c^{-1} & c & c & c^{-1} & c & 1 \end{bmatrix}, \quad (168)$$

$$\widetilde{C}_{13B}^{(0)} = \begin{bmatrix} 1 & d & d^{-1} & d & d & d^{-1} & d^{-1} & d^{-1} & d^{-1} & d & d & d^{-1} & d \\ d & 1 & d & d^{-1} & d & d & d^{-1} & d^{-1} & d^{-1} & d^{-1} & d & d & d^{-1} \\ d^{-1} & d & 1 & d & d^{-1} & d & d & d^{-1} & d^{-1} & d^{-1} & d^{-1} & d & d \\ d & d^{-1} & d & 1 & d & d^{-1} & d & d & d^{-1} & d^{-1} & d^{-1} & d^{-1} & d \\ d & d & d^{-1} & d & 1 & d & d^{-1} & d & d & d^{-1} & d^{-1} & d^{-1} & d^{-1} \\ d^{-1} & d & d & d^{-1} & d & 1 & d & d^{-1} & d & d & d^{-1} & d^{-1} & d^{-1} \\ d^{-1} & d^{-1} & d & d & d^{-1} & d & 1 & d & d^{-1} & d & d & d^{-1} & d^{-1} \\ d^{-1} & d^{-1} & d^{-1} & d & d & d^{-1} & d & 1 & d & d^{-1} & d & d & d^{-1} \\ d^{-1} & d^{-1} & d^{-1} & d^{-1} & d & d & d^{-1} & d & 1 & d & d^{-1} & d & d \\ d & d^{-1} & d^{-1} & d^{-1} & d^{-1} & d & d & d^{-1} & d & 1 & d & d^{-1} & d \\ d & d & d^{-1} & d^{-1} & d^{-1} & d^{-1} & d & d & d^{-1} & d & 1 & d & d^{-1} \\ d^{-1} & d & d & d^{-1} & d^{-1} & d^{-1} & d^{-1} & d & d & d^{-1} & d & 1 & d \\ d & d^{-1} & d & d & d^{-1} & d^{-1} & d^{-1} & d^{-1} & d & d & d^{-1} & d & 1 \end{bmatrix}, (169)$$

$$C_{13A}^{(0)} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & c^{-2} & c & c^{-3} & c^{-1} & c & c^{-1} & c^{-1} & c^{-1} & c^{-3} & c^{-1} & c & c^{-3} \\ 1 & c & c^2 & c & c^{-1} & c^3 & c^3 & c & c & c^{-1} & c^{-1} & c^3 & c \\ 1 & c^{-3} & c & c^{-2} & c^{-1} & c^{-1} & c & c & c^{-1} & c^{-3} & c^{-3} & c^{-1} & c^{-1} \\ 1 & c^{-1} & c^{-1} & c^{-1} & c^{-2} & c & c^{-1} & c & c & c^{-3} & c^{-3} & c^{-1} & c^{-3} \\ 1 & c & c^3 & c^{-1} & c & c^2 & c^3 & c & c^3 & c & c^{-1} & c & c^{-1} \\ 1 & c^{-1} & c^3 & c & c^{-1} & c^3 & c^2 & c^3 & c & c & c & c & c^{-1} \\ 1 & c^{-1} & c & c & c & c & c^3 & c^2 & c^3 & c^{-1} & c & c^3 & c^{-1} \\ 1 & c^{-1} & c & c^{-1} & c & c^3 & c & c^3 & c^2 & c & c^{-1} & c^3 & c \\ 1 & c^{-3} & c^{-1} & c^{-3} & c^{-3} & c & c & c^{-1} & c & c^{-2} & c^{-1} & c^{-1} & c^{-1} \\ 1 & c^{-1} & c^{-1} & c^{-3} & c^{-3} & c^{-1} & c & c & c^{-1} & c^{-1} & c^{-2} & c & c^{-3} \\ 1 & c & c^3 & c^{-1} & c^{-1} & c & c & c^3 & c^3 & c^{-1} & c & c^2 & c \\ 1 & c^{-3} & c & c^{-1} & c^{-3} & c^{-1} & c^{-1} & c^{-1} & c & c^{-1} & c^{-3} & c & c^{-2} \end{bmatrix}, \quad (170)$$

$$C_{13B}^{(0)} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & d^{-2} & d & d^{-3} & d^{-1} & d & d^{-1} & d^{-1} & d^{-1} & d^{-3} & d^{-1} & d & d^{-3} \\ 1 & d & d^2 & d & d^{-1} & d^3 & d^3 & d & d & d^{-1} & d^{-1} & d^3 & d \\ 1 & d^{-3} & d & d^{-2} & d^{-1} & d^{-1} & d & d & d^{-1} & d^{-3} & d^{-3} & d^{-1} & d^{-1} \\ 1 & d^{-1} & d^{-1} & d^{-1} & d^{-2} & d & d^{-1} & d & d & d^{-3} & d^{-3} & d^{-1} & d^{-3} \\ 1 & d & d^3 & d^{-1} & d & d^2 & d^3 & d & d^3 & d & d^{-1} & d & d^{-1} \\ 1 & d^{-1} & d^3 & d & d^{-1} & d^3 & d^2 & d^3 & d & d & d & d & d^{-1} \\ 1 & d^{-1} & d & d & d & d & d^3 & d^2 & d^3 & d^{-1} & d & d^3 & d^{-1} \\ 1 & d^{-1} & d & d^{-1} & d & d^3 & d & d^3 & d^2 & d & d^{-1} & d^3 & d \\ 1 & d^{-3} & d^{-1} & d^{-3} & d^{-3} & d & d & d^{-1} & d & d^{-2} & d^{-1} & d^{-1} & d^{-1} \\ 1 & d^{-1} & d^{-1} & d^{-3} & d^{-3} & d^{-1} & d & d & d^{-1} & d^{-1} & d^{-2} & d & d^{-3} \\ 1 & d & d^3 & d^{-1} & d^{-1} & d & d & d^3 & d^3 & d^{-1} & d & d^2 & d \\ 1 & d^{-3} & d & d^{-1} & d^{-3} & d^{-1} & d^{-1} & d^{-1} & d & d^{-1} & d^{-3} & d & d^{-2} \end{bmatrix}. \quad (171)$$

5.14. $N = 14$

5.14.1. *Orbits stemming from* F_{14}

The only maximal affine Hadamard families stemming from F_{14} are:

$$\begin{aligned} F_{14}^{(6)}(a,b,c,d,e,f) &= F_{14} \circ \mathrm{EXP}\left(\boldsymbol{i} \cdot R_{F_{14}^{(6)}}(a,b,c,d,e,f)\right), \qquad (172)\\ \left(F_{14}^{(6)}(a,b,c,d,e,f)\right)^T &= F_{14} \circ \mathrm{EXP}\left(\boldsymbol{i} \cdot \left(R_{F_{14}^{(6)}}(a,b,c,d,e,f)\right)^T\right), \end{aligned}$$

where

$$F_{14} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & w & w^2 & w^3 & w^4 & w^5 & w^6 & w^7 & w^8 & w^9 & w^{10} & w^{11} & w^{12} & w^{13} \\ 1 & w^2 & w^4 & w^6 & w^8 & w^{10} & w^{12} & 1 & w^2 & w^4 & w^6 & w^8 & w^{10} & w^{12} \\ 1 & w^3 & w^6 & w^9 & w^{12} & w & w^4 & w^7 & w^{10} & w^{13} & w^2 & w^5 & w^8 & w^{11} \\ 1 & w^4 & w^8 & w^{12} & w^2 & w^6 & w^{10} & 1 & w^4 & w^8 & w^{12} & w^2 & w^6 & w^{10} \\ 1 & w^5 & w^{10} & w & w^6 & w^{11} & w^2 & w^7 & w^{12} & w^3 & w^8 & w^{13} & w^4 & w^9 \\ 1 & w^6 & w^{12} & w^4 & w^{10} & w^2 & w^8 & 1 & w^6 & w^{12} & w^4 & w^{10} & w^2 & w^8 \\ 1 & w^7 & 1 & w^7 & 1 & w^7 & 1 & w^7 & 1 & w^7 & 1 & w^7 & 1 & w^7 \\ 1 & w^8 & w^2 & w^{10} & w^4 & w^{12} & w^6 & 1 & w^8 & w^2 & w^{10} & w^4 & w^{12} & w^6 \\ 1 & w^9 & w^4 & w^{13} & w^8 & w^3 & w^{12} & w^7 & w^2 & w^{11} & w^6 & w & w^{10} & w^5 \\ 1 & w^{10} & w^6 & w^2 & w^{12} & w^8 & w^4 & 1 & w^{10} & w^6 & w^2 & w^{12} & w^8 & w^4 \\ 1 & w^{11} & w^8 & w^5 & w^2 & w^{13} & w^{10} & w^7 & w^4 & w & w^{12} & w^9 & w^6 & w^3 \\ 1 & w^{12} & w^{10} & w^8 & w^6 & w^4 & w^2 & 1 & w^{12} & w^{10} & w^8 & w^6 & w^4 & w^2 \\ 1 & w^{13} & w^{12} & w^{11} & w^{10} & w^9 & w^8 & w^7 & w^6 & w^5 & w^4 & w^3 & w^2 & w \end{bmatrix}, \qquad (173)$$

with $w = \exp(\boldsymbol{i} \cdot 2\pi/14)$, so $w^{14} = 1$, $w^7 = -1$, and

$$R_{F_{14}^{(6)}}(a,b,c,d,e,f) = \left[\begin{array}{ccccccc|ccccccc} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & a & b & c & d & e & f & \bullet & a & b & c & d & e & f \\ \hline \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & a & b & c & d & e & f & \bullet & a & b & c & d & e & f \\ \hline \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & a & b & c & d & e & f & \bullet & a & b & c & d & e & f \\ \hline \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & a & b & c & d & e & f & \bullet & a & b & c & d & e & f \\ \hline \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & a & b & c & d & e & f & \bullet & a & b & c & d & e & f \\ \hline \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & a & b & c & d & e & f & \bullet & a & b & c & d & e & f \\ \hline \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & a & b & c & d & e & f & \bullet & a & b & c & d & e & f \end{array}\right]. \qquad (174)$$

$F_{14}^{(6)}$ and $\left(F_{14}^{(6)}\right)^T$ are a pair of cognate families. At least one of them can be obtained by permuting one of Diţă's constructions: either

$$\widetilde{F}_{14A}^{(6)}(\alpha_1,\ldots,\alpha_6) = \left[\begin{array}{c|c} [F_2]_{1,1} \cdot F_7 & [F_2]_{1,2} \cdot D(\alpha_1,\ldots,\alpha_6) \cdot F_7 \\ \hline [F_2]_{2,1} \cdot F_7 & [F_2]_{2,2} \cdot D(\alpha_1,\ldots,\alpha_6) \cdot F_7 \end{array}\right], \qquad (175)$$

where $D(\alpha_1,\ldots,\alpha_6)$ is the 7×7 diagonal matrix $\mathrm{diag}(1,e^{\boldsymbol{i}\alpha_1},\ldots,e^{\boldsymbol{i}\alpha_6})$, or

$$\widetilde{F}^{(6)}_{14B}(\beta_1,\ldots,\beta_6) \tag{176}$$

such that its i,j-th 2×2 block is equal to $[F_7]_{i,j}\cdot D(\alpha)\cdot F_2$, where $i,j\in\{1\ldots 7\}$, $D(\alpha)$ is the diagonal matrix $\mathrm{diag}(1,e^{\boldsymbol{i}\alpha})$ and $\alpha = 0,\beta_1,\ldots,\beta_6$ for $j=1,2,\ldots,7$ respectively.

This is because $\widetilde{F}^{(6)}_{14A}(\mathbf{0}) = F_2\otimes F_7$ and $\widetilde{F}^{(6)}_{14B}(\mathbf{0}) = F_7\otimes F_2$ are, according to [59], permutation equivalent to F_{14}, so both Diţă's orbits are maximal affine Hadamard families stemming from permuted F_{14}'s.

5.14.2. Other 14×14 *orbits*

Other dephased 14×14 orbits can be obtained, using the Diţă's method, from F_2 and dephased 7×7 complex Hadamard matrices from Sect. 5.7., for example:

$$FP^{(7)}_{14}(\alpha_1,\ldots,\alpha_7) = \left[\begin{array}{c|c} [F_2]_{1,1}\cdot F^{(0)}_7 & [F_2]_{1,2}\cdot D(\alpha_2,\ldots,\alpha_7)\cdot P^{(1)}_7(\alpha_1) \\ \hline [F_2]_{2,1}\cdot F^{(0)}_7 & [F_2]_{2,2}\cdot D(\alpha_2,\ldots,\alpha_7)\cdot P^{(1)}_7(\alpha_1) \end{array}\right],$$

$$FC^{(6)}_{14k}(\alpha_1,\ldots,\alpha_6) = \left[\begin{array}{c|c} [F_2]_{1,1}\cdot F^{(0)}_7 & [F_2]_{1,2}\cdot D(\alpha_1,\ldots,\alpha_6)\cdot C^{(0)}_{7k} \\ \hline [F_2]_{2,1}\cdot F^{(0)}_7 & [F_2]_{2,2}\cdot D(\alpha_1,\ldots,\alpha_6)\cdot C^{(0)}_{7k} \end{array}\right],$$

$$PP^{(8)}_{14}(\alpha_1,\ldots,\alpha_8) = \left[\begin{array}{c|c} [F_2]_{1,1}\cdot P^{(1)}_7(\alpha_1) & [F_2]_{1,2}\cdot D(\alpha_3,\ldots,\alpha_8)\cdot P^{(1)}_7(\alpha_2) \\ \hline [F_2]_{2,1}\cdot P^{(1)}_7(\alpha_1) & [F_2]_{2,2}\cdot D(\alpha_3,\ldots,\alpha_8)\cdot P^{(1)}_7(\alpha_2) \end{array}\right],$$

$$PC^{(7)}_{14k}(\alpha_1,\ldots,\alpha_7) = \left[\begin{array}{c|c} [F_2]_{1,1}\cdot P^{(1)}_7(\alpha_1) & [F_2]_{1,2}\cdot D(\alpha_2,\ldots,\alpha_7)\cdot C^{(0)}_{7k} \\ \hline [F_2]_{2,1}\cdot P^{(1)}_7(\alpha_1) & [F_2]_{2,2}\cdot D(\alpha_2,\ldots,\alpha_7)\cdot C^{(0)}_{7k} \end{array}\right],$$

$$CC^{(6)}_{14lm}(\alpha_1,\ldots,\alpha_6) = \left[\begin{array}{c|c} [F_2]_{1,1}\cdot C^{(0)}_{7l} & [F_2]_{1,2}\cdot D(\alpha_1,\ldots,\alpha_6)\cdot C^{(0)}_{7m} \\ \hline [F_2]_{2,1}\cdot C^{(0)}_{7l} & [F_2]_{2,2}\cdot D(\alpha_1,\ldots,\alpha_6)\cdot C^{(0)}_{7m} \end{array}\right],$$

where $k\in\{A,B,C,D\}$ and $(l,m)\in\{(A,A),(A,B),(A,C),(A,D),(B,B),(B,C),(B,D),(C,C),(C,D),(D,D)\}$ designate 7×7 complex Hadamard matrices associated with cyclic 7 roots, and $D(\beta_1,\ldots,\beta_6)$ denotes the 7×7 diagonal matrix $\mathrm{diag}(1,e^{\boldsymbol{i}\beta_1},\ldots,e^{\boldsymbol{i}\beta_6})$.

5.15. $N = 15$

5.15.1. *Orbits stemming from F_{15}*

The only maximal affine Hadamard families stemming from F_{15} are:

$$\begin{aligned} F_{15}^{(8)}(a,b,c,d,e,f,g,h) &= F_{15} \circ \mathrm{EXP}\left(\boldsymbol{i} \cdot R_{F_{15}^{(8)}}(a,b,c,d,e,f,g,h)\right), \quad (177) \\ \left(F_{15}^{(8)}(a,b,c,d,e,f,g,h)\right)^T &= F_{15} \circ \mathrm{EXP}\left(\boldsymbol{i} \cdot \left(R_{F_{15}^{(8)}}(a,b,c,d,e,f,g,h)\right)^T\right), \end{aligned}$$

where

$$F_{15} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & w & w^2 & w^3 & w^4 & w^5 & w^6 & w^7 & w^8 & w^9 & w^{10} & w^{11} & w^{12} & w^{13} & w^{14} \\ 1 & w^2 & w^4 & w^6 & w^8 & w^{10} & w^{12} & w^{14} & w & w^3 & w^5 & w^7 & w^9 & w^{11} & w^{13} \\ 1 & w^3 & w^6 & w^9 & w^{12} & 1 & w^3 & w^6 & w^9 & w^{12} & 1 & w^3 & w^6 & w^9 & w^{12} \\ 1 & w^4 & w^8 & w^{12} & w & w^5 & w^9 & w^{13} & w^2 & w^6 & w^{10} & w^{14} & w^3 & w^7 & w^{11} \\ 1 & w^5 & w^{10} & 1 & w^5 & w^{10} & 1 & w^5 & w^{10} & 1 & w^5 & w^{10} & 1 & w^5 & w^{10} \\ 1 & w^6 & w^{12} & w^3 & w^9 & 1 & w^6 & w^{12} & w^3 & w^9 & 1 & w^6 & w^{12} & w^3 & w^9 \\ 1 & w^7 & w^{14} & w^6 & w^{13} & w^5 & w^{12} & w^4 & w^{11} & w^3 & w^{10} & w^2 & w^9 & w & w^8 \\ 1 & w^8 & w & w^9 & w^2 & w^{10} & w^3 & w^{11} & w^4 & w^{12} & w^5 & w^{13} & w^6 & w^{14} & w^7 \\ 1 & w^9 & w^3 & w^{12} & w^6 & 1 & w^9 & w^3 & w^{12} & w^6 & 1 & w^9 & w^3 & w^{12} & w^6 \\ 1 & w^{10} & w^5 & 1 & w^{10} & w^5 & 1 & w^{10} & w^5 & 1 & w^{10} & w^5 & 1 & w^{10} & w^5 \\ 1 & w^{11} & w^7 & w^3 & w^{14} & w^{10} & w^6 & w^2 & w^{13} & w^9 & w^5 & w & w^{12} & w^8 & w^4 \\ 1 & w^{12} & w^9 & w^6 & w^3 & 1 & w^{12} & w^9 & w^6 & w^3 & 1 & w^{12} & w^9 & w^6 & w^3 \\ 1 & w^{13} & w^{11} & w^9 & w^7 & w^5 & w^3 & w & w^{14} & w^{12} & w^{10} & w^8 & w^6 & w^4 & w^2 \\ 1 & w^{14} & w^{13} & w^{12} & w^{11} & w^{10} & w^9 & w^8 & w^7 & w^6 & w^5 & w^4 & w^3 & w^2 & w \end{bmatrix} \quad (178)$$

with $w = \exp(\boldsymbol{i} \cdot 2\pi/15)$, so $w^{15} = 1$, and

$$R_{F_{15}^{(8)}}(a,b,c,d,e,f,g,h) = \left[\begin{array}{ccccc|ccccc|ccccc} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & a & b & c & d & \bullet & a & b & c & d & \bullet & a & b & c & d \\ \bullet & e & f & g & h & \bullet & e & f & g & h & \bullet & e & f & g & h \\ \hline \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & a & b & c & d & \bullet & a & b & c & d & \bullet & a & b & c & d \\ \bullet & e & f & g & h & \bullet & e & f & g & h & \bullet & e & f & g & h \\ \hline \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & a & b & c & d & \bullet & a & b & c & d & \bullet & a & b & c & d \\ \bullet & e & f & g & h & \bullet & e & f & g & h & \bullet & e & f & g & h \\ \hline \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & a & b & c & d & \bullet & a & b & c & d & \bullet & a & b & c & d \\ \bullet & e & f & g & h & \bullet & e & f & g & h & \bullet & e & f & g & h \\ \hline \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & a & b & c & d & \bullet & a & b & c & d & \bullet & a & b & c & d \\ \bullet & e & f & g & h & \bullet & e & f & g & h & \bullet & e & f & g & h \end{array}\right]. \quad (179)$$

The families $F_{15}^{(8)}$ and $\left(F_{15}^{(8)}\right)^T$ are cognate. At least one of them can be obtained by permuting one of Diţă's constructions: either

$$\widetilde{F}_{15A}^{(8)}(\alpha_1,\ldots,\alpha_8) = \left[\begin{array}{c|c|c} [F_3]_{1,1} \cdot F_5 & [F_3]_{1,2} \cdot D(\alpha_1,\ldots,\alpha_4) \cdot F_5 & [F_3]_{1,3} \cdot D(\alpha_5,\ldots,\alpha_8) \cdot F_5 \\ \hline [F_3]_{2,1} \cdot F_5 & [F_3]_{2,2} \cdot D(\alpha_1,\ldots,\alpha_4) \cdot F_5 & [F_3]_{2,3} \cdot D(\alpha_5,\ldots,\alpha_8) \cdot F_5 \\ \hline [F_3]_{3,1} \cdot F_5 & [F_3]_{3,2} \cdot D(\alpha_1,\ldots,\alpha_4) \cdot F_5 & [F_3]_{3,3} \cdot D(\alpha_5,\ldots,\alpha_8) \cdot F_5 \end{array}\right], \quad (180)$$

where $D(\alpha_1, \ldots, \alpha_4)$ is the 5×5 diagonal matrix $\mathrm{diag}\,(1, e^{\boldsymbol{i}\alpha_1}, \ldots, e^{\boldsymbol{i}\alpha_4})$, or

$$\widetilde{F}_{15B}^{(8)}(\beta_1, \ldots, \beta_8)\,, \tag{181}$$

such that its i, j-th 3×3 block is equal to $[F_5]_{i,j} \cdot D(\alpha, \beta) \cdot F_3$, where $i, j \in \{1 \ldots 5\}$, $D(\alpha, \beta)$ is the diagonal matrix $\mathrm{diag}\,(1, e^{\boldsymbol{i}\alpha}, e^{\boldsymbol{i}\beta})$ and

$$(\alpha, \beta) \;=\; (0,0),\; (\beta_1, \beta_2),\; (\beta_3, \beta_4),\; (\beta_5, \beta_6),\; (\beta_7, \beta_8)$$

for $j = 1, 2, \ldots, 5$ respectively. This is because $\widetilde{F}_{15A}^{(8)}(\mathbf{0}) = F_3 \otimes F_5$ and $\widetilde{F}_{15B}^{(8)}(\mathbf{0}) = F_5 \otimes F_3$ are, according to [59], permutation equivalent to F_{15}, so both Diţă's orbits are maximal affine Hadamard families stemming from permuted F_{15}'s.

5.16. $N = 16$

5.16.1. Orbits stemming from F_{16}

The only maximal affine Hadamard family stemming from F_{16} is the 17-parameter orbit

$$F_{16}^{(17)}(a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,r) = F_{16} \circ \mathrm{EXP}(\boldsymbol{i} \cdot R_{F_{16}^{(17)}}(a, \ldots, r)), \tag{182}$$

where

$$F_{16} = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & w & w^2 & w^3 & w^4 & w^5 & w^6 & w^7 & w^8 & w^9 & w^{10} & w^{11} & w^{12} & w^{13} & w^{14} & w^{15} \\
1 & w^2 & w^4 & w^6 & w^8 & w^{10} & w^{12} & w^{14} & 1 & w^2 & w^4 & w^6 & w^8 & w^{10} & w^{12} & w^{14} \\
1 & w^3 & w^6 & w^9 & w^{12} & w^{15} & w^2 & w^5 & w^8 & w^{11} & w^{14} & w & w^4 & w^7 & w^{10} & w^{13} \\
1 & w^4 & w^8 & w^{12} & 1 & w^4 & w^8 & w^{12} & 1 & w^4 & w^8 & w^{12} & 1 & w^4 & w^8 & w^{12} \\
1 & w^5 & w^{10} & w^{15} & w^4 & w^9 & w^{14} & w^3 & w^8 & w^{13} & w^2 & w^7 & w^{12} & w & w^6 & w^{11} \\
1 & w^6 & w^{12} & w^2 & w^8 & w^{14} & w^4 & w^{10} & 1 & w^6 & w^{12} & w^2 & w^8 & w^{14} & w^4 & w^{10} \\
1 & w^7 & w^{14} & w^5 & w^{12} & w^3 & w^{10} & w & w^8 & w^{15} & w^6 & w^{13} & w^4 & w^{11} & w^2 & w^9 \\
1 & w^8 & 1 & w^8 & 1 & w^8 & 1 & w^8 & 1 & w^8 & 1 & w^8 & 1 & w^8 & 1 & w^8 \\
1 & w^9 & w^2 & w^{11} & w^4 & w^{13} & w^6 & w^{15} & w^8 & w & w^{10} & w^3 & w^{12} & w^5 & w^{14} & w^7 \\
1 & w^{10} & w^4 & w^{14} & w^8 & w^2 & w^{12} & w^6 & 1 & w^{10} & w^4 & w^{14} & w^8 & w^2 & w^{12} & w^6 \\
1 & w^{11} & w^6 & w & w^{12} & w^7 & w^2 & w^{13} & w^8 & w^3 & w^{14} & w^9 & w^4 & w^{15} & w^{10} & w^5 \\
1 & w^{12} & w^8 & w^4 & 1 & w^{12} & w^8 & w^4 & 1 & w^{12} & w^8 & w^4 & 1 & w^{12} & w^8 & w^4 \\
1 & w^{13} & w^{10} & w^7 & w^4 & w & w^{14} & w^{11} & w^8 & w^5 & w^2 & w^{15} & w^{12} & w^9 & w^6 & w^3 \\
1 & w^{14} & w^{12} & w^{10} & w^8 & w^6 & w^4 & w^2 & 1 & w^{14} & w^{12} & w^{10} & w^8 & w^6 & w^4 & w^2 \\
1 & w^{15} & w^{14} & w^{13} & w^{12} & w^{11} & w^{10} & w^9 & w^8 & w^7 & w^6 & w^5 & w^4 & w^3 & w^2 & w
\end{bmatrix}, \tag{183}$$

with $w = \exp(\boldsymbol{i} \cdot 2\pi/16)$, so $w^{16} = 1$, $w^8 = -1$, $w^4 = \boldsymbol{i}$, and (where for typographic purposes we denote $e - a + k$ by

$$\left(\begin{array}{c} e - a \\ +k \end{array} \right),$$

so this is not a binomial coefficient)

$$R_{F_{16}^{(17)}}(a,\ldots,r) = \tag{184}$$

$$\left[\begin{array}{cccccccc|cccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & a & b & c & d & e & f & g & \bullet & a & b & c & d & e & f & g \\
\bullet & h & i & j & \bullet & h & i & j & \bullet & h & i & j & \bullet & h & i & j \\
\bullet & k & l & m & d & e-a+k & f-b+l & g-c+m & \bullet & k & l & m & d & e-a+k & f-b+l & g-c+m \\
\bullet & n & \bullet & n & \bullet & n & \bullet & n & \bullet & n & \bullet & n & \bullet & n & \bullet & n \\
\bullet & o & b & c-a+o & d & e-a+o & f & g-a+o & \bullet & o & b & c-a+o & d & e-a+o & f & g-a+o \\
\bullet & p & i & j-h+p & \bullet & p & i & j-h+p & \bullet & p & i & j-h+p & \bullet & p & i & j-h+p \\
\bullet & r & l & m-k+r & d & e-a+r & f-b+l & g-c+m+r-k & \bullet & r & l & m-k+r & d & e-a+r & f-b+l & g-c+m+r-k \\
\hline
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & a & b & c & d & e & f & g & \bullet & a & b & c & d & e & f & g \\
\bullet & h & i & j & \bullet & h & i & j & \bullet & h & i & j & \bullet & h & i & j \\
\bullet & k & l & m & d & e-a+k & f-b+l & g-c+m & \bullet & k & l & m & d & e-a+k & f-b+l & g-c+m \\
\bullet & n & \bullet & n & \bullet & n & \bullet & n & \bullet & n & \bullet & n & \bullet & n & \bullet & n \\
\bullet & o & b & c-a+o & d & e-a+o & f & g-a+o & \bullet & o & b & c-a+o & d & e-a+o & f & g-a+o \\
\bullet & p & i & j-h+p & \bullet & p & i & j-h+p & \bullet & p & i & j-h+p & \bullet & p & i & j-h+p \\
\bullet & r & l & m-k+r & d & e-a+r & f-b+l & g-c+m+r-k & \bullet & r & l & m-k+r & d & e-a+r & f-b+l & g-c+m+r-k
\end{array}\right].$$

The 17-dimensional family $F_{16}^{(17)}$ of $N = 16$ complex Hadamard matrices is self-cognate. Note that this dimensionality coincides with the defect, $d(F_{16}) = 17$, which follows from (19). The above orbit is permutation equivalent to the orbit constructed with the Diţă's method:

$$\widetilde{F}_{16}^{(17)}(\alpha_1,\ldots,\alpha_{17}) = \left[\begin{array}{c|c} [F_2]_{1,1}\cdot F_8^{(5)}(\alpha_1,\ldots,\alpha_5) & [F_2]_{1,2}\cdot D(\alpha_{11},\ldots,\alpha_{17})\cdot F_8^{(5)}(\alpha_6,\ldots,\alpha_{10}) \\ \hline [F_2]_{2,1}\cdot F_8^{(5)}(\alpha_1,\ldots,\alpha_5) & [F_2]_{2,2}\cdot D(\alpha_{11},\ldots,\alpha_{17})\cdot F_8^{(5)}(\alpha_6,\ldots,\alpha_{10}) \end{array}\right], \tag{185}$$

where the only maximal affine Hadamard family stemming from F_8: $F_8^{(5)}$ such that $F_8^{(5)}(\mathbf{0}) = F_8$, is given by (121), and $D(\alpha_{11},\ldots,\alpha_{17})$ is the 8×8 diagonal matrix $\mathrm{diag}(1, e^{i\alpha_{11}},\ldots,e^{i\alpha_{17}})$.

This is because

$$\begin{aligned} F_{16} \;=\; & \widetilde{F}_{16}^{(17)}(\mathbf{0},(1/16)2\pi,(2/16)2\pi,\ldots,(7/16)2\pi)\cdot \\ & [e_1,e_3,e_5,e_7,e_9,e_{11},e_{13},e_{15},e_2,e_4,e_6,e_8,e_{10},e_{12},e_{14},e_{16}]^T, \end{aligned} \tag{186}$$

where e_i denotes the i-th standard basis column vector, so indeed (185) generates the only maximal affine Hadamard family stemming from a permuted F_{16}.

The above orbit also passes through $\widetilde{F}_{16}^{(17)}(\mathbf{0}) = F_2 \otimes F_8$, as well as through permuted $F_2 \otimes F_2 \otimes F_4$ and $F_2 \otimes F_2 \otimes F_2 \otimes F_2$, since $\widetilde{F}_8^{(5)}$ of (124), or the permutation equivalent $F_8^{(5)}$ of (121), both pass through permuted $F_2 \otimes F_4$ and $F_2 \otimes F_2 \otimes F_2$. Note that all the tensor products F_{16}, $F_2 \otimes F_8$, $F_2 \otimes F_2 \otimes F_4$ and $F_2 \otimes F_2 \otimes F_2 \otimes F_2$ are inequivalent [59].

6. Closing Remarks

Let us summarize our work by proposing a set of dephased representatives $\mathcal{G}_N$ of equivalence classes of Hadamard matrices of size $N = 2, 3, 4, 5$, and by enumerating the sets from the sum of which one should be able to extract such a set of representatives for $N = 6, \ldots, 16$. The dots indicate that the existence of other equivalence classes cannot be excluded. For instance, one could look for new inequivalent families for composite N using the construction of Diţă [22] with permuted *some* of the component families of Hadamard matrices of smaller size.

We tend to believe that the compiled list is minimal in the sense that each family is necessary, since it contains at least some matrices not equivalent to all others. However, the presented orbits of Hadamard matrices may be (partially) equivalent, and equivalences may hold within families as well as between them.

In the list of complex Hadamard matrices presented below, let $\{X_N^{(d)}\}$ denote the set of elements of the family $X_N^{(d)}$.

$N = 2, \qquad \mathcal{G}_2 = \{F_2^{(0)}\}.$

$N = 3, \qquad \mathcal{G}_3 = \{F_3^{(0)}\}.$

$N = 4, \qquad \mathcal{G}_4 = \left\{F_4^{(1)}(a)\,;\, a \in [0, \pi)\right\}.$

$N = 5, \qquad \mathcal{G}_5 = \{F_5^{(0)}\}.$

$N = 6, \qquad \mathcal{G}_6 \subset \{F_6^{(2)}\} \cup \left\{\left(F_6^{(2)}\right)^T\right\} \cup \{D_6^{(1)}\} \cup \{C_6^{(0)}\} \cup \{S_6^{(0)}\} \cup \ldots$

$N = 7, \qquad \mathcal{G}_7 \subset \{F_7^{(0)}\} \cup \{P_7^{(1)}\} \cup \{C_{7A}^{(0)}\} \cup \{C_{7B}^{(0)}\} \cup \{C_{7C}^{(0)}\} \cup \{C_{7D}^{(0)}\} \cup \ldots$

$N = 8, \qquad \mathcal{G}_8 \subset \{F_8^{(5)}\} \cup \ldots$

$N = 9, \qquad \mathcal{G}_9 \subset \{F_9^{(4)}\} \cup \ldots$

$N = 10, \qquad \mathcal{G}_{10} \subset \{F_{10}^{(4)}\} \cup \left\{\left(F_{10}^{(4)}\right)^T\right\} \cup \ldots$

$N = 11, \qquad \mathcal{G}_{11} \subset \{F_{11}^{(0)}\} \cup \{C_{11A}^{(0)}\} \cup \{C_{11B}^{(0)}\} \cup \{N_{11}^{(0)}\} \cup \ldots$

$N = 12, \qquad \mathcal{G}_{12} \subset \{F_{12A}^{(9)}\} \cup \{F_{12B}^{(9)}\} \cup \{F_{12C}^{(9)}\} \cup \{F_{12D}^{(9)}\} \cup \left\{\left(F_{12B}^{(9)}\right)^T\right\} \cup$

$$\left\{\left(F_{12C}^{(9)}\right)^T\right\} \cup \left\{\left(F_{12D}^{(9)}\right)^T\right\} \cup \{FD_{12}^{(8)}\} \cup \{FC_{12}^{(7)}\} \cup \{FS_{12}^{(7)}\} \cup$$

$$\{DD_{12}^{(7)}\} \cup \{DC_{12}^{(6)}\} \cup \{DS_{12}^{(6)}\} \cup \{CC_{12}^{(5)}\} \cup \{CS_{12}^{(5)}\} \cup \{SS_{12}^{(5)}\} \cup \ldots$$

$$N = 13, \qquad \mathcal{G}_{13} \subset \{F_{13}^{(0)}\} \cup \{C_{13A}^{(0)}\} \cup \{C_{13B}^{(0)}\} \cup \{P_{13}^{(2)}\} \cup \ldots$$

$$N = 14, \qquad \mathcal{G}_{14} \subset \{F_{14}^{(6)}\} \cup \left\{\left(F_{14}^{(6)}\right)^T\right\} \cup \{FP_{14}^{(7)}\} \cup$$

$$\bigcup_{k\in\{A,B,C,D\}} \{FC_{14k}^{(6)}\} \cup \{PP_{14}^{(8)}\} \cup \bigcup_{k\in\{A,B,C,D\}} \{PC_{14k}^{(7)}\} \cup$$

$$\bigcup_{(l,m)\in\{(A,A),(A,B),(A,C),(A,D),(B,B),(B,C),(B,D),(C,C),(C,D),(D,D)\}} \{CC_{14lm}^{(6)}\} \cup \ldots$$

$$N = 15, \qquad \mathcal{G}_{15} \subset \{F_{15}^{(8)}\} \cup \left\{\left(F_{15}^{(8)}\right)^T\right\} \cup \ldots$$

$$N = 16, \qquad \mathcal{G}_{16} \subset \{F_{16}^{(17)}\} \cup \ldots$$

Note that the presented list of equivalence classes is complete only for $N = 2, 3, 4, 5$, while for $N \geq 6$ the full set of solutions remains unknown. The list of open questions could be rather long, but let us mention here some most relevant.

i) Check if there exist other inequivalent complex Hadamard matrices of size $N = 6$.
ii) Find the ranges of parameters of the existing $N = 6$ families such that all cases included are not equivalent.
iii) Check whether there exists a continuous family of complex Hadamard matrices for $N = 11$.
iv) Investigate, if all inequivalent real Hadamard matrices of size $N = 16, 20$ belong to continuous families or if some of them are isolated.
v) Find for which N there exist continuous families of complex Hadamard matrices which are not affine, and which are not contained in affine Hadamard families of a larger dimension.
vi) Find the dimensionalities of continuous orbits of inequivalent Hadamard matrices stemming from F_N if N is *not* a power of prime.

Problems analogous to i) − iv) are obviously open for higher dimensions. Thus a lot of work is still required to get a full understanding of the properties of the set of complex Hadamard matrices, even for one-digit dimensions. In spite of this fact we tend to believe that the above collection of matrices will be useful to tackle different physical problems, in particular these motivated by the theory of quantum information [43]. Interestingly, the dimension $N = 6$, the smallest product of two different primes, is the first case for which not all complex Hadamard matrices are known, as well as the simplest case for which the MUB problem remains open [54].

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