COLOUR TRIPLET-VALUED WAVELETS AND SPLINES

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Abstract

The concept of colour and multispectral image recognition connects all the topics we are considering. Colour (multispectral) image processing is investigated in this paper using an algebraic approach based on triplet numbers. In the algebraic approach, each image element is considered not as a 3D vector, but as a triplet number. The main goal of the paper is to show that triplet algebra can be used to solve colour image processing problems in a natural and effective manner. In this work we propose novel methods for wavelet transforms and splines implementation in colour space.

1. Introduction

The concept of colour and multispectral image recognition connects all the topics we are considering. In this paper, the term "multicomponent (multispectral, multicolour) image" is used for an image with more than one component. An RGB image is an example of a colour image having three separate image components R(red), G(green), and B(blue). Most of the colour image processing was done in the usual colour vector representations, such as *RGB*, *HSV*, *CIE* ..., etc.

Multispectral (multichannel) image processing is investigated in this paper using an algebraic approach based on hypercomplex numbers. This has proven to be more appropriate for multispectral-valued signals compared to traditional component–wise approaches. The suitability of such technique is often credited to the inherent correlation that exists between the spectral channels of an image.

In the algebraic approach, each image element is considered not as a mD vector, but as a mD hypercomplex number (m is the number of image spectral channels). Note that both these suppositions are only hypothesis. We have no biological evidence in the form of experiments that would verify that the brain actually uses any of the algebraic properties coming from the structures of vector spaces or hypercomplex algebras. We only know that we are able to recognize object in an invariant manner.

The main goal of the paper is to show that hypercomplex algebras can be used to solve image processing problems in multispectral environment in a natural and effective manner. We know that primates and animals with different evolutionary histories have colour visual systems with various dimensions. For example, the human brain uses 3D hypercomplex (triplet) numbers to recognize colour (RGB)–images and mantis shrimps use 10D multiplet numbers to recognize multicolour images. It is our aim to show that the use of hypercomplex algebras fits more naturally to the tasks of recognition of multicolour patterns than the usage of colour vector spaces. One can argue that it is conceivable that the nature has through evolution also learned to utilize useful properties of hypercomplex numbers. Thus, the visual cortex might have the ability to operate as a hypercomplex algebra computing device.

In this paper we focus our attention on 3D hypercomplex (triplet) numbers and colour wavelets and colour splines in triplet-valued space.

2. Triplet algebra and colour images

Triplet numbers were considered by De Morgan [1] and Greaves [5]. According to them the numbers of the form $\mathcal{C} = x+y\varepsilon_{col}+z\varepsilon_{col}^2$ are called *triplet* and form algebra [7]:

$$\mathcal{A}_3(R) = \mathcal{A}_3(\mathbf{R}|1, \varepsilon_{col}, \varepsilon_{col}^2) := \mathbf{R}1 + \mathbf{R}\varepsilon_{col} + \mathbf{R}\varepsilon_{col}^2, \quad \varepsilon_{col}^3 = 1, \tag{1}$$

which is called *triplet (color) algebra* and denoted by $\mathcal{A}_3^{col}$.

One can show that triplet algebra is the direct sum of real $\mathbf{R}$ and complex $\mathbf{C}$ fields

$$\mathcal{A}_3(\mathbf{R}) = R \cdot \mathbf{e}_{lu} + \mathbf{C} \cdot \mathbf{E}_{Ch}, \tag{2}$$

where e_{lu} and E_{Ch} are the "real" and the "complex" hyperimaginery units, respectively. Therefore, every triplet $\mathcal{C}$

is a linear combination $\mathcal{C} = a\mathbf{e}_{lu} + \mathbf{z}\mathbf{E}_{Ch}$ of the "scalar" and "complex" parts $a\mathbf{e}_{lu}$ and $\mathbf{z}\mathbf{E}_{Ch}$, respectively. The real numbers $a \in \mathbf{R}$ are called *intensity (luminance) numbers* and complex numbers $\mathbf{z} \in \mathbf{C}$ we will call *chromaticity numbers*.

A color image can be considered as a vector-valued ((R,G,B)-valued) function

$$\mathbf{f}_{col}(x,y) = f_R(x,y)\mathbf{i} + f_G(x,y)\mathbf{j} + f_B(x,y)\mathbf{k}. \quad (3)$$

Using color transformation to separate the color image onto two terms: 1D luminance (intensity) term and 2D chromaticity term (color information) [7]:

$$\mathbf{f}_{col}(x,y) = f_{lu}(x,y)\mathbf{e} + \mathbf{f}_{Ch}(x,y), \quad (4)$$

where $\mathbf{e} = (\mathbf{i}+\mathbf{j}+\mathbf{k})/\sqrt{3}$) $\langle \mathbf{e}, \mathbf{f}_{Ch}(x,y)\rangle = 0$. This is well–known user oriented HSV model of perceptual colour space.

The same result is obtained if we consider a color image as a triplet-valued function [7–12], [14–17]:

$$\begin{aligned} &\mathbf{f}_{col}(x,y) = \\ &= f_R(x,y)1_{col} + f_G(x,y)\varepsilon_{col} + f_B(x,y)\varepsilon_{col}^2 = \\ &= f_{lu}(x,y)\cdot \mathbf{e}_{lu} + \mathbf{f}_{Ch}(x,y)\cdot \mathbf{E}_{Ch}. \end{aligned} \quad (5)$$

Most of the data used in image processing are real-valued. Wavelets and splines were developed for the real-valued signals and then generalized on complex-valued ones. But usually processing techniques do not depend on real of complex essence of the data and do not provide the different ways of real and complex data. Meanwhile, the particular qualities of complex data can have a certain interest. In our case colour images are represented as triplet-valued functions (5). They form the space of colour images as the space of triplet-valued functions. In this space we define a new triplet-valued (colour) wavelets.

Intensity term $f_{lu}(x,y)$ of the colour image is real-valued and therefore can be processed by the ordinary (gray–level) wavelet methods. The chromatic term $\mathbf{f}_{Ch}(x,y)$ is complex-valued and it is necessary to define a modification of wavelets and splines in order to take into account the complex nature of the chroma.

3. Triplet-valued wavelets

Let $\psi^R(t)$ be an ordinal real-valued mother wavelet. We define the complex-valued wavelet as an analytic signal [2]:

$$\psi^{Ch}(t) = \psi^R(t) + i\psi^I(t) = \psi^R(t) + i\mathcal{H}[\psi^R(t)], \quad (6)$$

where $\mathcal{H}[\psi^R(t)]$ is the Hilbert transform of the real-valued mother wavelet. Now we construct triplet-valued (colour) mother wavelet by

$$\psi^{col}(t) = \psi^{lu}(t)\cdot \mathbf{e}_{lu} + \psi^{Ch}(t)\cdot \mathbf{E}_{Ch}, \quad (7)$$

where ψ^{lu} is a real-valued wavelet (luminance wavelet) and ψ^{Ch} is the complex-valued (chromatic) wavelet defined by (6).

Further, we define tripled-valued continuous and discrete wavelet transforms.

Let $\psi^R(t)$ be a real-valued mother wavelet and its scaled and shifted versions

$$\psi^R_{s,\tau}(t) = \frac{1}{\sqrt{|s|}}\psi^R\left(\frac{t-\tau}{s}\right), \quad s,\tau \in \mathbf{R}, s \neq 0$$

form the orthonormal basis of the space $L^2(\mathbf{R})$.

Let $\psi^I(t)$ be an imaginary part of complex-valued mother wavelet in (6). Then the complex-valued wavelet basis is defined as

$$\begin{aligned} &\psi^{Ch}_{s,\tau}(t) = \psi^R_{s,\tau}(t) + i\psi^I_{s,\tau}(t) = \\ &= \frac{1}{\sqrt{|s|}}\psi^R\left(\frac{t-\tau}{s}\right) + i\frac{1}{\sqrt{|s|}}\psi^I\left(\frac{t-\tau}{s}\right), \end{aligned} \quad (8)$$

where

$$\psi^I_{s,\tau}(t)] = \frac{1}{\sqrt{|s|}}\psi^I\left(\frac{t-\tau}{s}\right)$$

are scaled and shifted versions of $\psi^I(t)$.

Hence, the tripled-valued (colour) wavelet basis is defined as follows

$$\psi^{col}_{s,\tau}(t) = \psi^{lu}_{s,\tau}(t)\cdot \mathbf{e}_{lu} + \psi^{Ch}_{s,\tau}(t)\cdot \mathbf{E}_{Ch}, \quad (9)$$

where $\psi^{lu}_{s,\tau}(t)$ is a real-valued wavelet basis for luminance terms and $\psi^{Ch}_{s,\tau}(t)$ is a complex-valued wavelet basis for chromatic terms.

Finally we define 1D and 2D direct and inverse triplet-valued continuous wavelet transforms. The pair of 1D transforms has the following form:

$$F^{col}_{\text{CWT}}(s,\tau) = \frac{1}{\sqrt{|s|}}\int_{-\infty}^{+\infty} f^{col}(t)\left[\psi^{col}_{s,\tau}(t))\right]^* dt, \quad (10)$$

$$f^{col}(t) = \frac{1}{C_\psi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F^{col}_{\text{CWT}}(s,\tau)\psi^{col}_{s,\tau}(t)\frac{ds}{s^2}d\tau, \quad (11)$$

where

$$C_\psi = \int_{-\infty}^{+\infty}\frac{|\Psi^{col}(\omega)|^2}{|\omega|}d\omega$$

is a normalization factor and $\Psi^{col}(\omega)$ is the Fourier transform of the tripled-valued mother wavelet $\psi^{col}(t)$.

In 2D case we have another pair of the tripled-valued continuous wavelet transforms:

$$F^{col}_{\text{CWT}}(s_1,\tau_1;s_2,\tau_2) = \frac{1}{\sqrt{|s_1||s_2|}}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f^{col}(x,y)\times$$

$$\times \left[\psi^{col}_{s_1,\tau_1}(x))\right]^* \left[\psi^{col}_{s_1,\tau_1}(x))\right]^* dxdy, \qquad (12)$$

$$f^{col}(x,y) = \frac{1}{C_\psi^2} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F^{col}_{\text{CWT}}(s_1,\tau_1,s_2,\tau_2)\times$$

$$\times \psi^{col}_{s_1,\tau_1}(x,y)\psi^{col}_{s_2,\tau_2}(x,y)\frac{ds_1}{s_1^2}d\tau_1\frac{ds_2}{s_2^2}d\tau_2. \qquad (13)$$

Analogously, we define discrete counterpart of continuous wavelets. The real-valued wavelets have the following form

$$\psi^R_{n,k}(t) = 2^{-\frac{n}{2}}\psi^R(2^{-n}t-k),\ t\in \mathbf{R},\ n,k \in \mathbf{Z}^+, \quad (14)$$

where $\psi^R(t)$ is a mother wavelet. Thus, $\psi^R_{n,k}(t)$ is simply a scaling and translation of the mother wavelet, $\psi^R(t)$.

Using complex wavelet definition (6) we can define the complex-valued (chromatic) wavelet basis functions by

$$\psi^{Ch}_{n,k}(t) = \psi^R_{n,k}(t) + i\psi^I_{n,k}(t), \qquad (15)$$

where $\psi^I_{n,k}(t)$ is a scaling and translation of imaginary part of the complex mother wavelet as in Eq. (14). They form colour wavelet basis in the space of colour triplet-valued functions.

Finally, we construct triplet-valued (colour) wavelets by

$$\psi^{col}_{n,k} = \psi^{lu}_{n,k}\cdot \mathbf{e}_{lu} + \psi^{Ch}_{n,k}\cdot \mathbf{E}_{Ch}. \qquad (16)$$

They form colour wavelet basis in the space of colour triplet-valued functions.

Now, we can define discrete colour wavelet transforms. In 1D case the discrete colour wavelet transform has the following form:

$$F^{col}_{\text{DWT}}(n,k) = \int f^{col}(t)\left[\psi^{col}_{n,k}(t))\right]^* dt, \qquad (17)$$

$$f^{col}(t) = C_\psi \sum_n\sum_k F^{col}_{\text{DWT}}(n,k)\psi^{col}_{n,k}(t). \qquad (18)$$

and in 2D case we have

$$F^{col}_{\text{DWT}}(n_1,k_1;n_2,k_2) =$$

$$= \int\int f_{col}(x,y)\left[\psi^{col}_{n_1,k_1}(x)\right]^*\left[\psi^{col}_{n_2,k_2}(y)\right]^* dxdy, \quad (19)$$

$$f^{col}(x,y) = C_\psi^2 \sum_{n_1}\sum_{k_1}\sum_{n_2}\sum_{k_2} F^{col}_{\text{DWT}}(n_1,k_1;n_2,k_2)\times$$

$$\times \psi^{col}_{n_1,k_1}(x,y)\psi^{col}_{n_2,k_2}(x,y). \qquad (20)$$

The real and imaginary parts of the different complex mother wavelets are shown in Fig. 1 and Fig. 2. Examples of chromatic and triplet-valued wavelets are shown in Fig. 4.

4. Triplet-valued splines

Similarly to colour wavelets, it is possible to construct colour splines.

Let $\beta^n(x)$ be a B-spline [18]. B-splines are symmetrical, bell shaped functions constructed from a rectangular pulse:

$$\beta_0(x) = \begin{cases} 1, & -\frac{1}{2} < x < \frac{1}{2} \\ \frac{1}{2}, & |x| = -\frac{1}{2} \\ 0, & \text{otherwise} \end{cases} \qquad (21)$$

by

$$\beta_n(x) = \frac{1}{n!}\sum_{k=0}^{n+1}\binom{n+1}{k}(-1)^k\left(x-k+\frac{n+1}{2}\right)^n_+, \qquad (22)$$

where $(x)^n_+ = \max(0,x^n)$. We define the complex-valued B-spline as an analytic signal:

$$\beta^{Ch}_n(x) = \beta^R_n(x) + i\beta^I_n(x) = \beta^R_n(x) + i\mathcal{H}[\beta^R_n(x)], \ (23)$$

where $\mathcal{H}[\beta^R_n(x)]$ is the Hilbert transform of real-valued B-spline $\beta^R_n(x)$. We construct triplet-valued (colour) B-splines in the same way as triplet-valued wavelets:

$$\beta^{col}_n(x) = \beta_n(x)\cdot \mathbf{e}_{lu} + \beta^{Ch}_n(x)\cdot \mathbf{E}_{Ch}. \qquad (24)$$

Another colour spline types (for example, Moms, Keys', Schaum's, sinc modifications) can be constructed in the same way. First, construct the complex-valued spline as an analytic signal $s^{Ch}(x))$ of spline function $s(x)$. Second, construct the tripled-valued spline as $s^{col}(x) = s(x)\cdot \mathbf{e}_{lu} + s^{Ch}(x)\cdot \mathbf{E}_{Ch}$.

The real and imaginary parts of the different complex splines are shown in Fig. 3 and examples of chromatic and triplet-valued splines are shown in Fig. 5.

5. Conclusion

In this study we define a new method of constructing triplet-valued (colour)) wavelets and splines. It is based on triplet algebra and Hilbert transform of signal.

References

[1] A. De Morgan. On the foundation of algebra. *Trans. Cambridge Philos. Soc.*, 8(3):241–254, 1847.

[2] C. L. Du and W. L. Hwang. *Singularity detection and characterization with complex-valued wavelets and their applications*. Technical Report TR-IIS-00-010, Academia Sinica Institute of Information Science, Taiwan, 2000.

[3] M. I. Fomitchev. Introduction into wavelets. *Matematicheskaya Morfologiya*, 3(5), November 1998.

[4] A. Graps. An introduction to wavelets. *IEEE Computational Science and Engineering*, 2(2), Summer 1995.

[5] C. Greaves. On algebraic triplets. *Procn. Iris. Acad.*, 3:51–54, 57–64, 80–84, 105–108, 1847.

[6] H. Hamdan, H. Tsou, and S. M. Hinedi. *High–Dynamic Frequency Tracking Using Wavelets.* TDA Progress Report 42-128, Jet Propulson Laboratory, Pasadena, California, USA, February 1997.

[7] E. Labounets-Rundblad. *Fast Fourier–Clifford Transforms Design and Applications in Invariant Image Recofnition.* Tampere University of Technology, Tampere, Finland, 2000.

[8] E. V. Labunets, V. G. Labunets, K. Egiazarian, and J. Astola. Hypercomplex moments application in invariant image recognition. *International Conference on Image Processing*, pages 257–261, October 1998.

[9] V. G. Labunets, E. V. Labunets-Rundblad, and J. Astola. Algebra and geometry of colour images. *Proc. of First International Workshop on Spectral Tecniques and Logic Design for Future Digital Systems, Tampere, Finland*, pages 231–361, June 2000.

[10] V. G. Labunets, A. V. Maidan, J. Astola, and K. Egiazarian. Quazi–optimal Zernike feature selection. *IASTED Inter. Conf. Signal Processing and Communications SPC 2000*, pages 433–438, September 2000.

[11] E. V. Labunets-Rundblad and V. G. Labunets. *Chapter 7. Spatial–Colour Clifford Algebra for Invariant Image Recognition.* In: Geometric Computing with Clifford Algebra, Edt. G. Sommer., Springer, Berlin Heideberg, 2001.

[12] E. V. Labunets-Rundblad, V. G. Labunets, and J. Astola. Is the visual cortex a "Fast Clifford algebra quantum compüer"? *A NATO Advanced Research Workshop "Clifford Analysis and Its Applications"*, October 1999, be published.

[13] L. V. Novikov. *Fundamentals of wavelet analysis of signals (In Russian).* IAnII RAS, St. Peterburg, Russia, 1999.

[14] E. Rundblad, V. Labunets, J. Astola, K. Egiazarian, and S. Polovnev. Fast invariant recognition of colour images based on Triplet–Fourier–Gauss transform. *Proc. of Conf. Computer Science and Information Technologies, Yerevan, Armenia*, pages 265–268, 1999.

[15] E. Rundblad, V. G. Labunets, and J. Astola. Fast calculation algorithms of invariants for color and multispectral image recognition. *Algebraic Frames for the Perception–Action Cycle. Second Inter. Workshop, AFPAC 2000*, September 2000.

[16] E. Rundblad, V. G. Labunets, K. Egiazarian, and J. Astola. Invariant recognition of color images based on Fourier–Clifford number theoretical transform. *EUROPORTO, Conf. on Image and Signal Processing for Remote Sensing VI*, September 2000, be published.

[17] E. Rundblad-Labunets and V. Labunets. Fast invariant recognition of multicolour images based on Multiplet–Fourier–Gauss transform. *Second International Workshop on Transform and Filter Banks, Tampere, Finland, TICSP Series*, (3):405–438, March 1999.

[18] M. Unser. Splines: A perfect fit for signal and image processing. *IEEE Signal Processing Magazine*, 16(6):22–38, November 1999.

[19] M. Vetterli and J. Kovačević. *Wavelets and Subband Coding.* Prentice Hall, Englewood Cliffs, New Jersey, 1995.

[20] V. I. Vorobyov and V. G. Gribunin. *Theory and practice of wavelet transform (In Russian).* VUS, St. Peterburg, Russia, 1999.

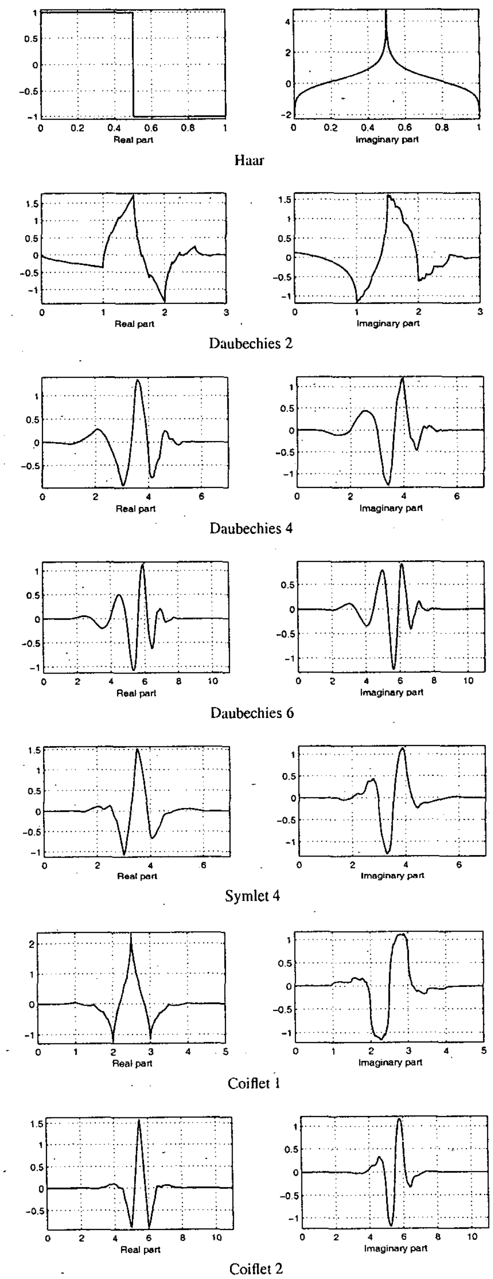

Figure 1. Examples of real and imaginary parts for different complex wavelets

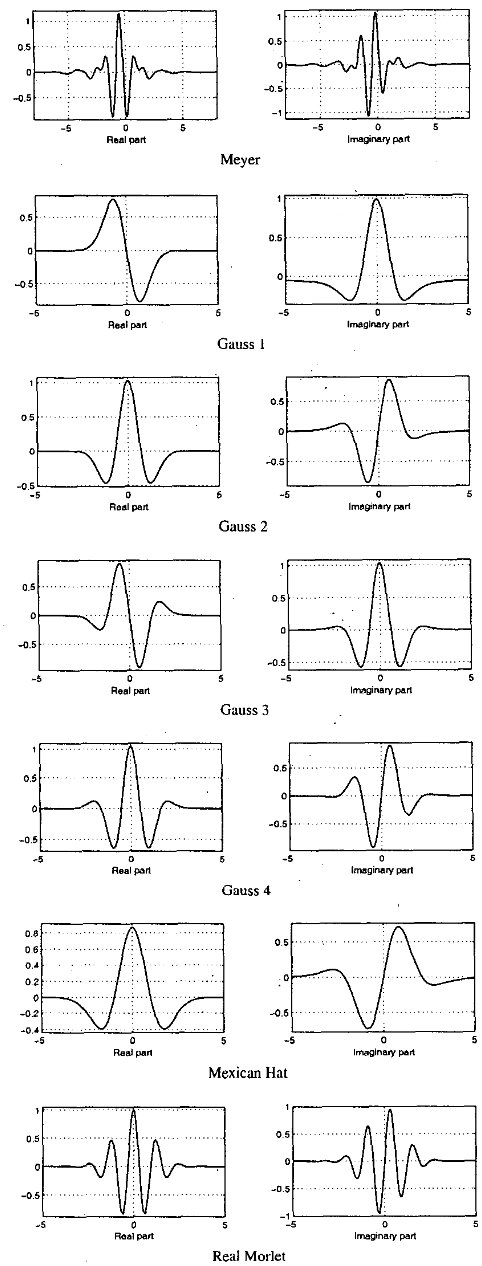

Figure 2. Examples of real and imaginary parts for different complex wavelets (continue)

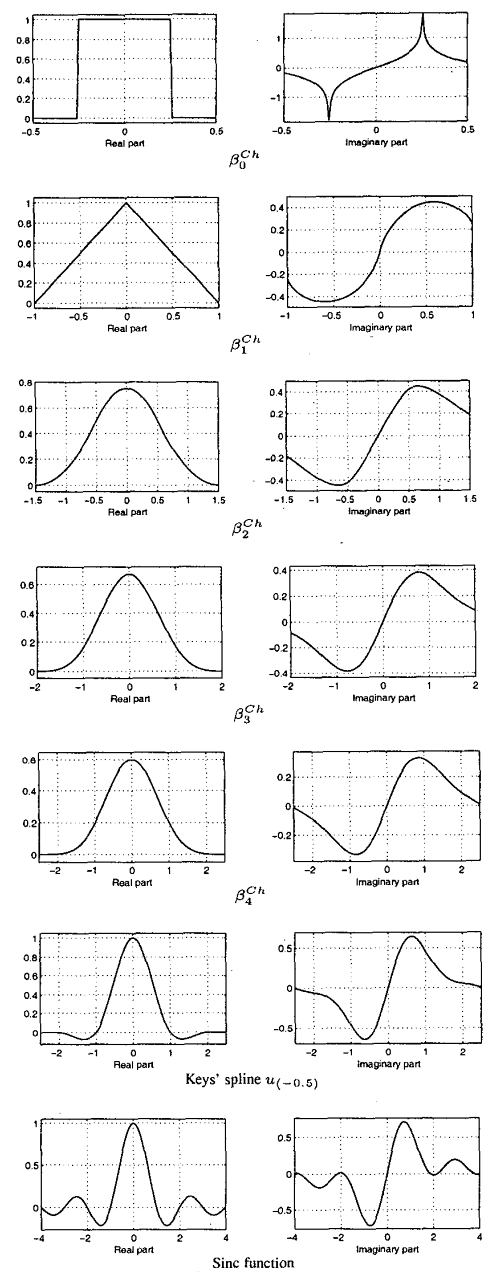

Figure 3. Examples of real and imaginary parts for different complex splines

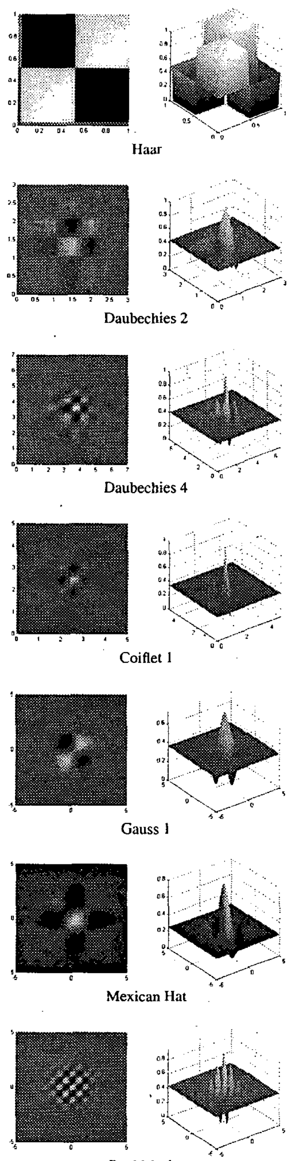

Figure 4. Left: examples of chromatic wavelets (intensity is equal to 0.5); Right: examples of triplet-valued wavelets as a function two spatial coordinates (third axis is intensity, surface is coloured in accordance with chromatic component values)

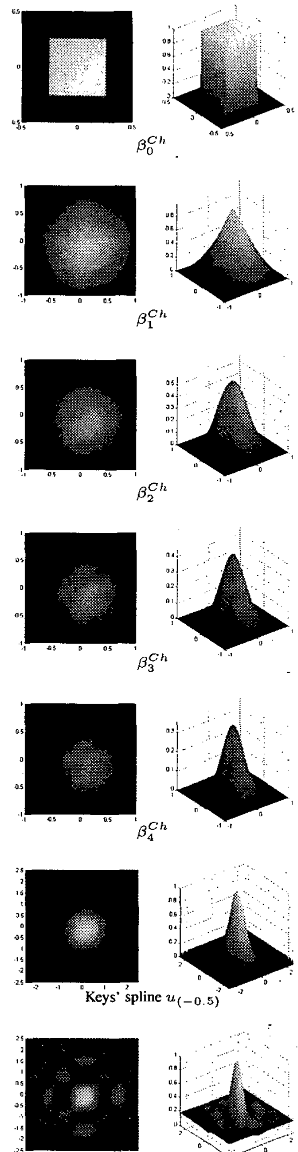

Figure 5. Left: examples of chromatic splines (intensity is equal to 0.5); Right: examples of triplet-valued splines as a function two spatial coordinates (third axis is intensity, surface is coloured in accordance with chromatic component values)