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Family of Unified Complex Hadamard Transforms

Susanto Rahardja and Bogdan J. Falkowski

***Abstract*— Novel discrete orthogonal transforms are introduced in this paper, namely the unified complex Hadamard transforms. These transforms have elements confined to four elementary complex integer numbers which are generated based on the Walsh–Hadamard transform, using a single unifying mathematical formula. The generation of higher dimension transformation matrices are discussed in detail.**

***Index Terms*—Digital signal processing, discrete transforms, fast algorithms, orthogonal transforms, unified complex Hadamard transforms.**

I. Introduction

In this brief, novel discrete orthogonal transformations with elements which are integer-valued complex numbers and may be considered as systems of complex Walsh functions are introduced. These transforms may be useful in applications where the need for complex-valued discrete orthogonal transforms arises, such as digital signal processing (DSP). These systems of functions and transformations are called complex Hadamard transforms (CHT's) and are confined to four complex values (± 1 and $\pm i$). In the literature, there exists another transformation based on four-valued complex Walsh functions, called the "complex BIFORE transform" [22]. For real-valued input data, the complex BIFORE transform reduces to a BIFORE or Hadamard transform whose bases are Walsh functions. The basic definition of the complex BIFORE transform is based on a recursive formula defining one class of complex Hadamard matrices that involves diagonalization of higher order matrices and multiple Kronecker products. The unified complex Hadamard transforms (UCHT's) have recently been considered as a tool in spectral approach to logic design [8], [9], [20], [21]. Like its predecessors, the UCHT's show similar properties and characteristics. The classical techniques of transforming will be employed, i.e., the truth vector of the function is transformed into a unique spectral domain, and by the fact that the transform matrix is orthogonal, the property of revealing some information more clearly while concealing others is sustained. The spectral domain is called a complex spectral domain, where the information from the truth vector is transformed, and divided into the real spectral domain and imaginary spectral domain. Both real and imaginary spectral domains are integer numbers. In order to increase the number of possible transformations for integer-valued matrices, Perkowski introduced the concept of linearly independent logic and polynomial expansions confined to operations in $\mathrm{GF}(2)$ [18], [19]. Fast transforms for Perkowski linearly independent logic were developed in [10]. The idea of using complex-valued rather than integer-valued transformation matrices for spectral processing of Boolean functions is considered for the first time in this article. By increasing still further the number of possible different entries in the transformation matrices with complex numbers, one can expect the reduction of their spectral representation, especially if both the original functions and their spectra are presented in the form of some kind of decision diagrams, which have already been proposed for UCHT's [8], [20]. In particular, the Walsh–Hadamard transform is one of many UCHT matrices introduced here. Some of the UCHT matrices have a unique half-spectrum property (HSP). There are general fast algorithms from the representation of transform matrices in the form of layered Kronecker matrices. In addition, constant-geometry fast algorithms with in-place architecture are also available for the new transforms. The complex BIFORE transform instead has only fast transform without constant geometry algorithm. The existence of constant-geometry fast butterflies is suitable for efficient very large-scale integration (VLSI) implementation. The introduced UCHT's may be used for various applications, where the Walsh-Hadamard transform has already been used [1], [2], [4]–[7], [11], [12], [17], [25], [26], [28], [31]–[39]. Generally, the UCHT's may be classified as the integer-valued and complex integer-valued transforms. The integer-valued and complex integer-valued matrices have elements confined to two (± 1) and four complex numbers (± 1 and $\pm i$), respectively. Comparing the complex integer-valued UCHT's between themselves, those that possess HSP will be advantageous as they require half of the spectral coefficients for analysis. However, it should be pointed out that if the functional data are real numbers, the existence of the HSP in complex integer-valued UCHT's has no additional storage advantage compared to the integer-valued counterparts (e.g., Walsh-Hadamard transform). But, the complex integer-valued transforms [3], [12], [23], [24], [26], [27] are suitable for problems with complex-valued functions and for such functions, the UCHT's with half spectrum property is the most compact representation.

A number of applications of new transforms in the area of spectral computer-aided design of digital circuit is shown in [8], [9], [20], [21]. Some of them are detection of Boolean symmetries and compact classification scheme [20], [21]. The above applications can be also performed using other spectral approaches based on Walsh and Reed–Muller transforms [7], [29] as well as classical approaches [13]. They are important in many real life problems of designing and optimizing digital circuits such as Boolean matching, technology mapping and designing with universal logic modules [7]. As mentioned earlier, some UCHT's are simply systems of complex Walsh functions while others become q-valued Chrestenson functions for $q = 2$ or 4 [26], [33]. It is then obvious that the UCHT's can be used in different applications of complex Walsh functions and Chrestenson functions in processing of multiple-valued functions, especially for the case of four-valued functions. Much work has already been done for Chrestenson transform, e.g., characterization of ternary threshold functions [15], development of measures of the dependence of multiple-valued logic functions on the linear logic

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functions [16], and disjoint spectral translation that allows extending the possibility of low complexity realization to a large class of multiple-valued logical functions [14]. Similar results can be obtained for these UCHT's that are different from Chrestenson functions. The methods should be computationally more effective as Chrestenson transform matrix does not possess HSP. Finally, based on UCHT's, new complex decision diagrams to store and calculate the UCHT spectra have been introduced by the same authors in [8], [20]. These new decision diagrams are complex hybrid decision diagrams, complex algebraic decision diagrams, complex multiterminal decision diagrams, real and imaginary decision diagrams and complex edge-valued decision diagrams. With the existence of HSP, the size of all different complex decision diagrams is always reduced by half. Since this reduction is unique only to UCHT's, it is obvious that such a feature is useful particularly in handling complex-valued data functions.

II. Family of Unified Complex Hadamard Transforms

In this section, the concept of a *family of UCHT's* is introduced and such a family is proposed to represent discrete signals and systems. Several properties of the new family are outlined. The generation of the transformation matrices commences from the basic Walsh-Hadamard transformation matrix. All members of UCHT's may be produced by newly defined direct matrix operator and recursively generated to higher dimension matrices by a single Kronecker product. It must be noted that although the basis functions in the definition that generates all UCHT matrices are discrete Walsh functions, each member of the newly defined UCHT fulfills requirements of complex Hadamard matrices; there are altogether 64 such different matrices that are introduced in this section, all of which are generated by one unifying formula. Half of the UCHT matrices fulfill the requirement of the novel half-spectrum theorem. The theorem shows that from the knowledge of only half the vector of the full spectrum, one is able to recover the full original data.

It is shown in this brief that UCHT matrices can be generated recursively in a variety of ways by using new operators on matrices. They are: the direct matrix operator, the $\bmod - q$ Kronecker operator, and the rotation operator. Different mathematical properties of such operators are given. The introduced UCHT's have several DSP applications. When dealing with complex signals or multiple-valued logic systems coded as complex numbers, there are some inherent computational advantages in UCHT matrices and can be used to efficiently analyze and synthesize such complex input functions.

III. Preliminaries

Definition 1: Let H be an $r \times c$ matrix, such that $[H] = \{h(j,k)\},\ 0 \le j \le r-1, 0 \le k \le c-1, h(j,k)$ is an element of H at row j and column k. Then the power matrix of an integer a with respect to H is an $r \times c$ matrix defined by

$$[a^H] = \{a^{h(j,k)}\}. \tag{1}$$

Definition 2: Let $[A] = \{a(j_A,k_A)\}$ be an $r_A \times c_A$ matrix and $[B] = \{b(j_B,k_B)\}$ be an $r_B \times c_B$ matrix, with $0 \le j_A \le r_A - 1, 0 \le k_A \le c_A - 1, 0 \le j_B \le r_B - 1$ and $0 \le k_B \le c_B - 1$. $a(j_A,k_A)$ and $b(j_B,k_B)$ are elements of A and B at row j_A, column k_A and row j_B, column k_B, respectively. The *mod-q Kronecker addition* of A and B denoted as $A\Theta_p B$, is defined by the following matrix operation, where elements in A are expanded in a Kronecker product manner into a submatrix with dimension $r_B \times c_B$ and the values in such a submatrix are the results of the particular element of $A \bmod - q$ addition with the respective elements of B, i.e.,

$$A\Theta_q B = \begin{bmatrix} a'(0,0) & a'(0,1) & \cdots & a'(0,c_A) \\ a'(1,0) & a'(1,1) & \cdots & a'(1,c_A) \\ \vdots & \vdots & \ddots & \vdots \\ a'(r_A,0) & a'(r_A,1) & \cdots & a'(r_A,c_A) \end{bmatrix} \tag{2}$$

and where we have the equation shown at the bottom of the page, with $+_q$ representing the $\bmod - q$ addition.

Definition 3: Let A be an $r \times c$ matrix, such that $[A] = \{a(j,k)\}$ where $0 \le j \le r, 0 \le k \le c, a(j,k)$ is an element of A at row j and column k. If $[B] = \{b(k)\}$ is a $1 \times c$ row matrix and $[C] = \{c(j)\}$ is an $r \times 1$ column matrix, then the matrix operator $\diamond$ is defined as

$$[A \diamond B] = \{a(j,k) \quad b(k)\} \tag{3}$$

and

$$[A \diamond C] = \{a(j,k) \quad c(j)\}. \tag{4}$$

Property 1: The following properties of $\diamond$ may be derived:

$$(A \diamond B)\otimes^n = (A\otimes^n) \diamond (B\otimes^n) \tag{5}$$
$$(AB^T) \diamond C = (A \diamond C)B^T \tag{6}$$
$$(A \diamond B_1) \diamond B_2 = A \diamond (B_1 \diamond B_2) \tag{7}$$
$$(A \diamond B \diamond C)^T = (A \diamond B)^T \diamond C^T \tag{8}$$

and

$$\overline{(A \diamond B \diamond C)} = \overline{(A \diamond B)} \diamond \bar{C} = \bar{A} \diamond \bar{B} \diamond \bar{C} \tag{9}$$

where $[B_1] = \{b_1(k)\}$ and $B_2 = \{b_2(k)\}$ are $1 \times c$ row matrices, and $\otimes^n$ denotes a right-hand side Kronecker product applied n times.

IV. Basic Definitions and Properties of UCHTs

In the definitions of existing discrete orthogonal transforms, the elements of transformation matrices normally consist of discrete values of $+1$ and -1, or generalizations that permit values of $e^{2\pi ni/q}$ for a prime q, which leads to a complete orthonormal system known as the Chrestenson system [26], [33]. In this section, some new matrix definitions are introduced to open a new concept of a family of discrete transforms that can be used to process complex and multiple-valued functions.

Through this brief, two sets $\mathcal{Z}_q$ and $\mathcal{C}_4$ are defined as $\mathcal{Z}_q = \{0, 1, \cdots, q-1\}$ and $\mathcal{C}_4 = \{+1, -1, +i, -i\}$, respectively. A is an $N \times N$ square matrix, where $[A] = \{a(j,k)\}, a(j,k) \in \mathcal{Z}_q$.

Definition 4: If $H_n = i^A$ is a square nonsingular matrix of order $N = q^n$, there exists a unique inverse H_n^{-1}. If $[H_n]$ and $[H_n^{-1}]$ have elements from $\mathcal{C}_4$ then H_n is a multiple-valued transform.

The transformation matrices are defined by a set of basis discrete-valued functions. To ensure that no information is lost in the resulting spectrum, orthogonality in the transformation matrix is essential. This

$$a'(j_A,k_A) = \begin{bmatrix} a(j_A,k_A) +_q b(0,0) & a(j_A,k_A) +_q b(0,1) & \cdots & a(j_A,k_A) +_q b(0,c_B) \\ a(j_A,k_A) +_q b(1,0) & a(j_A,k_A) +_q b(1,1) & \cdots & a(j_A,k_A) +_q b(1,c_B) \\ \vdots & \vdots & \ddots & \vdots \\ a(j_A,k_A) +_q b(r_B,0) & a(j_A,k_A) +_q b(r_B,1) & \cdots & a(j_A,k_A) +_q b(r_B,c_B) \end{bmatrix}$$

requires zero correlation between pairs of different basis functions. In general, if H is an orthogonal $N \times N$ matrix with real entries, then

$$HH^T = NI. \tag{10}$$

Definition 5: Let $q = 4$ and H represent the resulting matrix of $H = i^A = \{h(j,k)\}$, then H is said to be orthogonal in the complex domain [30] if

$$|\det H| = N^{1/2N} \tag{11}$$

and

$$HH^* = H^*H = NI \tag{12}$$

where H^* represents the complex conjugate transpose of H, and H is said to be a CHT. The resulting matrix H can be easily used as a binary, ternary, or quaternary transform as any two, three, or all four elements in the transformation matrix can be used for coding of two-, three-, or four-valued logic functions respectively. In addition, with an appropriate coding of the original function, the UCHT may be used as a multiple-valued transform.

Definition 6: The transformation core matrix for any UCHT is defined as

$$H_1^{[\tau]} = W_1 \diamond \begin{bmatrix}\tau_1 \\ \tau_2\end{bmatrix} \diamond [1 \quad \tau_3] \tag{13}$$

where W_1 is the Walsh–Hadamard transform matrix of order 2 [9], [22]

$$\tau = \sum_{j=1}^{3} 4^{3-j} \log_i \tau_j, \quad \tau_j \in \mathcal{C}_4, \quad j \in \{1,2,3\}.$$

Equation (13) may be expressed in a matrix form as

$$H_1^{[\tau]} \equiv A_L^{[\tau]} W_1 \equiv W_1 A_R^{[\tau]} \tag{14}$$

where

$$A_L^{[\tau]} = \frac{1}{2}\begin{bmatrix}\tau_1(1+\tau_3) & \tau_1(1-\tau_3) \\ \tau_2(1-\tau_3) & \tau_2(1+\tau_3)\end{bmatrix} \tag{15}$$

and

$$A_R^{[\tau]} = \frac{1}{2}\begin{bmatrix}\tau_1+\tau_2 & \tau_3(\tau_1-\tau_2) \\ \tau_1-\tau_2 & \tau_3(\tau_1+\tau_2)\end{bmatrix} \tag{16}$$

and the italic subscripts L and R denote the left and right matrix, respectively, as shown in (14). The proofs of (15) and (16) are immediate from properties of UCHT. From (13), it is obvious that there are, altogether, $4^3 = 64$ UCHT's.

All UCHT matrices can be separated into two groups of 32 basic matrices dependent on the existence of the HSP (Theorem 5). These transformation matrices are listed in Table I. The symbol "$\surd$" indicates the existence of the half-spectrum property for a given transformation matrix.

Theorem 1: Let A be a 2×2 multiple-valued matrix $[A] = \{a(j,k)\}, a(j,k) \in \mathcal{Z}_4$. Let $H_1^{[\tau]} = i^A$ represent the mapping of 4-valued integers into unit circle of complex plane with elements in $\mathcal{C}_4$, where H_1 is the smallest (core) matrix of the size 2×2. Then the condition of existence of the UCHT matrix for $H_1^{[\tau]}$ is

$$|a(0,0) + a(1,1) - a(0,1) - a(1,0)| = 2. \tag{17}$$

Proof: From (13) of Definition 6 and since $H_1^{[\tau]} = i^A$

$$a(0,0) = \log_i \tau_1, \quad a(0,1) = \log_i \tau_1\tau_3$$
$$a(1,0) = \log_i \tau_2, \quad a(1,1) = \log_i -\tau_2\tau_3.$$

Solving the four equations, (17) is proved. □

Property 2: If $H_1^{[\tau]} = i^A$ and the condition of existence of UCHT matrix is satisfied, then the higher dimension matrix H_n of size $2^n \times 2^n$ is orthogonal in complex domain if

$$H_n = i^{A_n} \tag{18}$$

where $A_n = A\Theta_q \overset{n \text{ times}}{\cdots\cdots\cdots\cdots} \Theta_q A$ and $H_n^{-1} = 1/NH_n$.

Definition 7: Let H_n be a $2^n \times 2^n$ square matrix, such that H_n is recursively defined by

$$H_n = \begin{bmatrix}H_{n-1}^{(1)} & H_{n-1}^{(2)} \\ H_{n-1}^{(3)} & H_{n-1}^{(4)}\end{bmatrix} \tag{19}$$

where each submatrix $H_{n-1}^{(j)}, \; j \in \{1,2,3,4\}$, has dimension of $2^{n-1} \times 2^{n-1}$. The *Rotation* operator R on the square matrix H_n is recursively defined as 4^{n-r} clockwise rotations involving 4^{n-r+1} submatrices each of order 2^{r-1} for $r = n, n-1, \cdots, 2, 1$.

Definition 8: The *Inverse Rotation* operator R^{-1} on a $2^n \times 2^n$ square matrix H_n is recursively defined as 4^{n-r} anticlockwise rotations involving 4^{n-r+1} submatrices each of order 2^{r-1} for $r = n, n-1, \cdots, 2, 1$.

Property 3: Let $H_1^{[\tau]}$ be defined for some τ. Then if

$$H_1^{[\tau]} = \begin{bmatrix}h_0^{[\tau]} & h_1^{[\tau]} \\ h_2^{[\tau]} & h_3^{[\tau]}\end{bmatrix} \tag{20}$$

$h_j^{[\tau]} \in \mathcal{C}_4, j \in \mathcal{Z}_4, \log_i(h_j^{[\tau]}) \in \mathcal{Z}_4$ and $H_1^{[\tau]}$ is orthogonal in the complex domain, then the higher dimension matrix $H_n^{[\tau]}$ of the size $2^n \times 2^n$ is orthogonal in complex domain if

$$H_n^{[\tau]} = \begin{bmatrix}R^{-\log_i(h_0^{[\tau]})}\left(H_{n-1}^{[\tau]}\right) & R^{-\log_i(h_1^{[\tau]})}\left(H_{n-1}^{[\tau]}\right) \\ R^{-\log_i(h_2^{[\tau]})}\left(H_{n-1}^{[\tau]}\right) & R^{-\log_i(h_3^{[\tau]})}\left(H_{n-1}^{[\tau]}\right)\end{bmatrix} \tag{21}$$

where R is the Rotation operator on a recursive matrix, which is given by Definitions 7, 8, and (20), respectively.

Property 4: Let Property 3 be satisfied for a multiple-valued matrix A, where $[A] = \{a(j,k)\}, a(j,k) \in \mathcal{Z}_4$. Then, for any τ

$$H_n = \begin{bmatrix}R^{-a(0,0)}H_{n-1} & R^{-a(0,1)}H_{n-1} \\ R^{-a(1,0)}H_{n-1} & R^{-a(1,1)}H_{n-1}\end{bmatrix}. \tag{22}$$

Theorem 2: Let $H_n^{[\tau]}$ be any UCHT matrix. If $h(j,k)$ is an element of $H_n^{[\tau]}$ at row j and column k, where $0 \le j,k \le 2^n$, then

$$h(j,k) = \prod_{r=0}^{n-1} h_r \tag{23}$$

where

$$h_r = \tau_1 + (\tau_2 - \tau_1)j_r + \tau_1(\tau_3 - 1)k_r + (\tau_1 - \tau_1\tau_3 - \tau_2 - \tau_2\tau_3)j_r k_r. \tag{24}$$

$\langle j_{n-1}, j_{n-2}, \cdots, j_0\rangle$ and $\langle k_{n-1}, k_{n-2}, \cdots, k_0\rangle$ denote the respective binary representations of the decimals j and k, respectively, i.e., $\langle j\rangle_{10} = \langle j_{n-1}, j_{n-2}, \cdots, j_0\rangle_2$ and $\langle k\rangle_{10} = \langle k_{n-1}, k_{n-2}, \cdots, k_0\rangle_2$.

Proof: From (13) of Definition 6, $h(0,0) = \tau_1, h(0,1) = \tau_1\tau_3, h(1,0) = \tau_2$ and $h(1,1) = -\tau_2\tau_3$. If $n = 1$, then by the arithmetic expansion [2], h_0 may be written as $h_0 = \tau_1 + (\tau_2 - \tau_1)j_0 + \tau_1(\tau_3 - 1)k_0 + (\tau_1 - \tau_1\tau_3 - \tau_2 - \tau_2\tau_3)j_0k_0$.

Since, from Property 2, $H_1^{[\tau]}$ is derived from the power matrix of the unit complex number i with respect to multiple *mod-q Kronecker addition* of the respective multiple-valued matrix A, any of the elements of $H_n^{[\tau]}$ denoted as $h(j,k)$ is derived from n times multiplication of each corresponding h_r with r ranging from 0 to $n-1$. Hence

$$h(j,k) = \prod_{r=0}^{n-1} h_r.$$

□

TABLE I
LIST OF UCHT'S **HSP**

$\tau_1 = 1$				$\tau_1 = i$				$\tau_1 = -1$				$\tau_1 = -i$			
τ_2	τ_3	*HSP*	UCHT Matrix	τ_2	τ_3	*HSP*	UCHT Matrix	τ_2	τ_3	*HSP*	UCHT Matrix	τ_2	τ_3	*HSP*	UCHT Matrix
1	1		$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$	1	1		$\begin{pmatrix} i & i \\ 1 & -1 \end{pmatrix}$	1	1		$\begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$	1	1		$\begin{pmatrix} -i & -i \\ 1 & -1 \end{pmatrix}$
1	i	,	$\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$	1	i	,	$\begin{pmatrix} i & -1 \\ 1 & -i \end{pmatrix}$	1	i	,	$\begin{pmatrix} -1 & -i \\ 1 & -i \end{pmatrix}$	1	i	,	$\begin{pmatrix} -i & 1 \\ 1 & -i \end{pmatrix}$
1	-1		$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$	1	-1		$\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$	1	-1		$\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$	1	-1		$\begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}$
1	$-i$	,	$\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$	1	$-i$	,	$\begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}$	1	$-i$	,	$\begin{pmatrix} -1 & i \\ 1 & i \end{pmatrix}$	1	$-i$	,	$\begin{pmatrix} -i & -1 \\ 1 & i \end{pmatrix}$
i	1		$\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$	i	1		$\begin{pmatrix} i & i \\ i & -i \end{pmatrix}$	i	1		$\begin{pmatrix} -1 & -1 \\ i & -i \end{pmatrix}$	i	1		$\begin{pmatrix} -i & -i \\ i & -i \end{pmatrix}$
i	i	,	$\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$	i	i	,	$\begin{pmatrix} i & -1 \\ i & 1 \end{pmatrix}$	i	i	,	$\begin{pmatrix} -1 & -i \\ i & 1 \end{pmatrix}$	i	i	,	$\begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix}$
i	-1		$\begin{pmatrix} 1 & -1 \\ i & i \end{pmatrix}$	i	-1		$\begin{pmatrix} i & -i \\ i & i \end{pmatrix}$	i	-1		$\begin{pmatrix} -1 & 1 \\ i & i \end{pmatrix}$	i	-1		$\begin{pmatrix} -i & i \\ i & i \end{pmatrix}$
i	$-i$	,	$\begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix}$	i	$-i$	,	$\begin{pmatrix} i & 1 \\ i & -1 \end{pmatrix}$	i	$-i$	,	$\begin{pmatrix} -1 & i \\ i & -1 \end{pmatrix}$	i	$-i$	,	$\begin{pmatrix} -i & -1 \\ i & -1 \end{pmatrix}$
-1	1		$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$	-1	1		$\begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix}$	-1	1		$\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$	-1	1		$\begin{pmatrix} -i & -i \\ -1 & 1 \end{pmatrix}$
-1	i	,	$\begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix}$	-1	i	,	$\begin{pmatrix} i & -1 \\ -1 & i \end{pmatrix}$	-1	i	,	$\begin{pmatrix} -1 & -i \\ -1 & i \end{pmatrix}$	-1	i	,	$\begin{pmatrix} -i & 1 \\ -1 & i \end{pmatrix}$
-1	-1		$\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$	-1	-1		$\begin{pmatrix} i & -i \\ -1 & -1 \end{pmatrix}$	-1	-1		$\begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$	-1	-1		$\begin{pmatrix} -i & -i \\ -1 & -1 \end{pmatrix}$
-1	$-i$	,	$\begin{pmatrix} 1 & -i \\ -1 & -i \end{pmatrix}$	-1	$-i$	,	$\begin{pmatrix} i & 1 \\ -1 & -i \end{pmatrix}$	-1	$-i$	,	$\begin{pmatrix} -1 & i \\ -1 & -i \end{pmatrix}$	-1	$-i$	,	$\begin{pmatrix} -i & -1 \\ -1 & -i \end{pmatrix}$
$-i$	1		$\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$	$-i$	1		$\begin{pmatrix} i & i \\ -i & i \end{pmatrix}$	$-i$	1		$\begin{pmatrix} -1 & -1 \\ -i & i \end{pmatrix}$	$-i$	1		$\begin{pmatrix} -i & -i \\ -i & i \end{pmatrix}$
$-i$	i	,	$\begin{pmatrix} 1 & i \\ -i & -1 \end{pmatrix}$	$-i$	i	,	$\begin{pmatrix} i & -1 \\ -i & -1 \end{pmatrix}$	$-i$	i	,	$\begin{pmatrix} -1 & -i \\ -i & -1 \end{pmatrix}$	$-i$	i	,	$\begin{pmatrix} -i & 1 \\ -i & -1 \end{pmatrix}$
$-i$	-1		$\begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix}$	$-i$	-1		$\begin{pmatrix} i & -i \\ -i & -i \end{pmatrix}$	$-i$	-1		$\begin{pmatrix} -1 & 1 \\ -i & -i \end{pmatrix}$	$-i$	-1		$\begin{pmatrix} -i & i \\ -i & -i \end{pmatrix}$
$-i$	$-i$	,	$\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$	$-i$	$-i$	,	$\begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix}$	$-i$	$-i$	,	$\begin{pmatrix} -1 & i \\ -i & 1 \end{pmatrix}$	$-i$	$-i$	,	$\begin{pmatrix} -i & -1 \\ -i & 1 \end{pmatrix}$

Lemma 1: For $\tau = 0$ in (13)

$$h(j,k) = w(j,k) = (-1)^{\sum_{r=0}^{n-1} j_r k_r} \tag{25}$$

where $w(j,k)$ defines the element of the Walsh–Hadamard transform, W_n at row j and column k, respectively.

Proof: Let $\tau = 0$ in (13). Then, $\tau_1 = \tau_2 = \tau_3 = 1$. Substituting, we have $H_1^{[\tau]} \equiv W_1$. From (23) and (24)

$$h(j,k) = \prod_{r=0}^{n-1} 1 - 2j_r k_r.$$

But $(-1)^{jk} = 1 - 2jk$, hence

$$h(j,k) = (-1)^{\sum\limits_{r=0}^{n-1} j_r k_r} \equiv w(j,k).$$

□

Lemma 1 shows that the Walsh–Hadamard transform is simply one of many UCHT's derived from Definition 6.

Definition 9: Let $F(j)$ denote the coded data sequence where $0 \le j \le N-1$ and $F(j)$ may be a real or complex number which depends on a particular coding of the data. Then, like other discrete transforms, the corresponding UCHT of the data sequence may be expressed in the matrix form as

$$\vec{Z} = H_n \vec{F} \tag{26}$$

where $\vec{Z} = [Z(0), Z(1), \cdots, Z(k), \cdots, Z(N-1)]^T$ and $\vec{F} = [F(0), F(1), \cdots, F(j), \cdots, F(N-1)]^T$. The values of $Z(k)$ are complex numbers. Since the transform is orthogonal, the data sequence may be uniquely recovered by the inverse transform, i.e.,

$$\vec{F} = H_n^{-1} \vec{Z} = \frac{1}{N} H_n \vec{Z}. \tag{27}$$

Expressed in the form of one-dimensional discrete Fourier transform (DFT), (26) and (27) are

$$Z(k) = \sum_{j=0}^{N-1} h(k,j)F(j) = \sum_{j=0}^{N-1} F(j)\left(\prod_{r=0}^{n-1} h_r\right) \tag{28}$$

and

$$F(j) = \frac{1}{N}\sum_{k=0}^{N-1} Z(k)\left(\prod_{r=0}^{n-1} h_r\right) \tag{29}$$

where h_r is defined as in (24) with the indices j and k interchanged.

Definition 10: Let H be an orthogonal transform in the real or complex domain. By Definition 9, a spectrum of integer-valued data sequence may be represented by (26) and (27). The existence of *HSP* for the transform matrix is defined as the existence of a $(1/2)N \times 1$ vector $\vec{Z}_{1/2}$ which completely characterizes the full transform vector $\vec{Z}$ of the orthogonal transform, with the ability to recover the unique transform vector $\vec{Z}$.

Theorem 3: If a HSP is to exist in $H_1^{[\tau]}$, then τ_3 in (13) must be *imaginary*.

Proof: For the HSP to exist in transform vector $\vec{Z}$, nonlinear manipulation of rows in $H_1^{[\tau]}$ must be available to distinguish one row from the other. From Definition 6, if $\vec{H}_0$ and $\vec{H}_1$ represent 2×1 column vectors, such that

$$H_1^{[\tau]} = \begin{bmatrix} \vec{H}_0^T \\ \vec{H}_1^T \end{bmatrix}$$

then the obvious condition for such existence is $\vec{H}_1 = \lambda \bar{\vec{H}}_0$, where λ is a constant. Since $H_1^{[\tau]}$ is orthogonal in the complex domain, the rows are not linearly dependent of each other. Therefore, the existence of λ in the equation will determine the existence of the HSP. Therefore

$$\begin{pmatrix} \tau_2 \\ -\tau_2\tau_3 \end{pmatrix} = \lambda\overline{\begin{pmatrix} \tau_1 \\ \tau_1\tau_3 \end{pmatrix}} = \begin{pmatrix} \lambda\bar{\tau}_1 \\ \lambda\overline{\tau_1\tau_3} \end{pmatrix} = \begin{pmatrix} \lambda\bar{\tau}_1 \\ \lambda\bar{\tau}_1\bar{\tau}_3 \end{pmatrix}.$$

Hence, $\tau_3 = -\bar{\tau}_3$. Since $\tau_3 \in \mathcal{C}_4$, Theorem 3 is proved. □

Theorem 3 shows that the HSP exists for 32 UCHT's.

Theorem 4: If $H_n^{[\tau]}$ satisfies Theorem 3, then the elements of the UCHT matrix are mathematically related by

$$h(N-1-j,k) = (\tau_1\tau_2)^\gamma \overline{h(j,k)} \tag{30}$$

where $\gamma = n \bmod 4$.

Proof: From (23) and (24)

$$h(j,k) = \prod_{r=0}^{n-1}\Big[\tau_1 + (\tau_2 - \tau_1)j_r + \tau_1(\tau_3 - 1)k_r + (\tau_1 - \tau_1\tau_3 - \tau_2 - \tau_2\tau_3)j_r k_r\Big]$$

where $0 \leq j,k \leq 2^n, j_r$ and k_r denote the r-th bit binary representations of the decimal j and k respectively. Then

$$\begin{aligned} h(N-1-j,k) &= \prod_{r=0}^{n-1}[\tau_1 + (\tau_2 - \tau_1)\bar{j}_r + \tau_1(\tau_3 - 1)k_r \\ &\quad + (\tau_1 - \tau_1\tau_3 - \tau_2 - \tau_2\tau_3)\bar{j}_r k_r] \\ &\equiv \prod_{r=0}^{n-1} \alpha + \beta\bar{j}_r \end{aligned}$$

where α and β are two linear functions of k_r, such that $\alpha = \tau_1 + \tau_1(\tau_3 - 1)k_r$ and $\beta = (\tau_2 - \tau_1) + (\tau_1 - \tau_2 - \tau_1\tau_3 - \tau_2\tau_3)k_r$. Then

$$\overline{h(j,k)} = \prod_{r=0}^{n-1}(\bar{\alpha} + \bar{\beta}j_r).$$

It follows that

$$\begin{aligned} \frac{h(N-1-j,k)}{\overline{h(j,k)}} &= \prod_{r=0}^{n-1}\left(\frac{\alpha + \beta\bar{j}_r}{\bar{\alpha} + \bar{\beta}j_r}\right) \\ &= \prod_{r=0}^{n-1}\left(\frac{\alpha+\beta}{\bar{\alpha}}\bar{j}_r + \frac{\alpha}{\bar{\alpha}+\bar{\beta}}j_r\right) \end{aligned}$$

and

$$\begin{aligned} &\frac{\alpha+\beta}{\bar{\alpha}} \\ &\quad = \frac{\tau_1 + \tau_1(\tau_3 - 1)k_r + (\tau_2 - \tau_1) + (\tau_1 - \tau_2 - \tau_1\tau_3 - \tau_2\tau_3)k_r}{\bar{\tau}_1 + \bar{\tau}_1(\bar{\tau}_3 - 1)k_r} \\ &\quad = \frac{\tau_2 - \tau_2(\tau_3 + 1)k_r}{\bar{\tau}_1 + \bar{\tau}_1(\bar{\tau}_3 - 1)k_r}. \end{aligned}$$

Similarly

$$\begin{aligned} &\frac{\alpha}{\bar{\alpha}+\bar{\beta}} \\ &\quad = \frac{\tau_1 + \tau_1(\tau_3 - 1)k_r}{\bar{\tau}_1 + \bar{\tau}_1(\bar{\tau}_3 - 1)k_r + (\bar{\tau}_2 - \bar{\tau}_1) + (\bar{\tau}_1 - \bar{\tau}_2 - \bar{\tau}_1\bar{\tau}_3 - \bar{\tau}_2\bar{\tau}_3)k_r} \\ &\quad = \frac{\tau_1 + \tau_1(\tau_3 - 1)k_r}{\bar{\tau}_2 - \bar{\tau}_2(\bar{\tau}_3 + 1)k_r}. \end{aligned}$$

If τ_3 is imaginary and $\tau_3 \in \mathcal{C}_4$, then the equations reduce to

$$\frac{\alpha+\beta}{\bar{\alpha}} = \frac{\tau_2}{\bar{\tau}_1}$$

and

$$\frac{\alpha}{\bar{\alpha}+\bar{\beta}} = \frac{\tau_1}{\bar{\tau}_2}$$

respectively. Since $\tau_1, \tau_2 \in \mathcal{C}_4$, therefore $\tau_1\bar{\tau}_1 = \tau_2\bar{\tau}_2 = 1$. Then

$$\frac{h(N-1-j,k)}{\overline{h(j,k)}} = \prod_{r=0}^{n-1}[(\tau_1\tau_2)\bar{j}_r + (\tau_1\tau_2)j_r] = (\tau_1\tau_2)^n.$$

Since $(\tau_1\tau_2) \in \mathcal{C}_4, \mathcal{C}_4$ is a set with four complex unitary elements, and the integer power of any element in $\mathcal{C}_4$ is cyclic, then if $\gamma = n \bmod 4, (\tau_1\tau_2)^n \equiv (\tau_1\tau_2)^\gamma$. The proof of (30) is completed. □

Theorem 5: If $H_n^{[\tau]}$ satisfies Theorem 3 and $\vec{F}$ and $\vec{Z}$ are according to Definition 9, then

$$Z(N-1-k) = (\tau_1\tau_2)^\gamma \overline{Z(k)} \tag{31}$$

where $\gamma = n \bmod 4$.

Proof: From (28)

$$Z(k) = \sum_{j=0}^{N-1} h(k,j)F(j).$$

Then

$$Z(N-1-k) = \sum_{j=0}^{N-1} h(N-1-k,j)F(j).$$

From (30)

$$Z(N-1-k) = \sum_{j=0}^{N-1}[(\tau_1\tau_2)^\gamma \overline{h(k,j)}]F(j).$$

Then

$$Z(N-1-k) = (\tau_1\tau_2)^\gamma \sum_{j=0}^{N-1}\overline{h(k,j)}F(j).$$

This completes the proof of (31). □

Theorem 5 is called the half-spectrum theorem, which implies that only half of the spectral coefficients are required for synthesis and analysis. This will reduce the computational cost of the UCHT by half. If the signal is purely real valued, the integer-valued UCHT's (e.g., Walsh–Hadamard transforms) are better choices, since the complex integer-valued counterparts (though they use only half the number of spectral coefficients) each requires double storage. The existence of HSP in the complex integer-valued UCHT's is particularly useful in the synthesis and analysis of complex-valued signals or multiple-valued data, where each discrete data is coded into some complex integer. With such a coding, the resulting spectral coefficients will also be complex integers. Most signals in radar, sonar, and communications have in-phase and quadrature components, i.e., they are complex valued [5]. Hence, there is a practical need to operate on complex numbers.

Lemma 2: Let A be a 2×2 multiple-valued matrix, $[A] = \{a(j,k)\}, a(j,k) \in \mathcal{Z}_4$. Let $H_1 = i^A$ represent the mapping of 4-valued integers into the unit circle of the complex plane. Then, the higher dimension matrix H_n of size $2^n \times 2^n$ is

$$H_n = H_1 \otimes H_{n-1} = H_{n-1} \otimes H_1 = H_1 \otimes^n = i^A \otimes^n = i^{A_n} \tag{32}$$

where $\otimes$ denotes the Kronecker product and $\otimes^n$ represents the n-time multiple Kronecker products. Also, by Definition 6

$$H_n^{[\tau]} = \left[W_1 \diamond \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} \diamond [1 \quad \tau_3] \right] \otimes^n \tag{33}$$

where $N = 2^n$. Equation (5) may be applied to (33) to reduce the computational time.

V. Conclusion

A new class of discrete orthogonal transforms has been introduced. The transform is based on the mapping of 4-valued integers into the unit circle of the complex plane with elements strictly in the set $\{1, -1, i, -i\}$. Under the various permutations of the integers, there exist some conditions which will lead to the transform being mapped to and being orthogonal in the complex domain. This has been identified as the family of UCHT's, as the mapping of the multiple-valued transforms into the complex domain will result in square basis matrices which satisfy the Hadamard's determinant equation in the complex domain. Intuitively, Walsh–Hadamard being an integer-valued transform is merely a special case of the UCHT's.

Nonlinear manipulations of the transform matrices lead to the HSP. This property does not exist for all the UCHT's. Conditions for the existence of the property have been derived and proven. The existence has caused a reduction in computational costs of those special transforms, and leads to the derivation of compact representation of multiple- and complex-valued functions [8], [20]. This property does not hold for the well-known Walsh–Hadamard transform.

Another advantage of the new transform is the existence of not only fast algorithms based on layered Kronecker products that can be represented by a series of strand matrices (which is similar to the complex BIFORE transform), but also a constant geometry fast algorithm that is well suited to VLSI hardware implementation. In such an architecture, only one butterfly stage has to be implemented and the processed data can be fed back to the input to be processed by the same circuitry.

Signal parameters in many DSP applications are estimated using the Fourier power spectrum. However, computing the Fourier transform is relatively complicated and there are applications for which it is important to achieve hardware savings, even at the expense of some loss in parameter estimation accuracy, as is the case in satellite radar altimetry. The Walsh–Hadamard transform is used for such an application [5]. Also, it is well known from the literature that the fast Walsh–Hadamard transform can be efficiently used for the calculation of the DFT [28] for implementing adaptive filters [11] and for DFT spectrum filter realizations [34]. The usual frequency-domain FIR filtering problem can be easily converted into a Walsh frequency-domain filtering problem, and similar structure results in a possible alternative for infinite-impulse response filter implementations [17]. An efficient method for implementation of a class of isotropic quadratic filters using the Walsh–Hadamard transform was also proposed [6]. Advantages of the 2-D Walsh–Hadamard transform, also known as S or sequential transform [25], in lossless image compression are well known. An integrated-circuit chip implementing 2-D Walsh–Hadamard transform has been implemented for commercial applications by Philips Corporation [25]. Some other applications of Walsh–Hadamard and other related transforms are described in [1], [2], [4], [12], [26], [31]–[33]. As the Walsh–Hadamard transform is one of the UCHT's, it is thus believed that the important properties of the UCHT's presented in this article may also be of interest to researchers working in the above-mentioned areas where the standard Walsh–Hadamard matrices had been applied.

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Data Clustering Using Hierarchical Deterministic Annealing and Higher Order Statistics

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***Abstract*—In this brief, we propose an extension to the hierarchical deterministic annealing (HDA) algorithm for clustering by incorporating additional features into the algorithm. To decide a split in a cluster, the interdependency among all the clusters is taken into account by using the entire data distribution. A general distortion measure derived from the higher order statistics (HOS) of the data is used to analyze the phase transitions. Experimental results clearly demonstrate the improvement in the performance of the HDA algorithm when the interdependency among the clusters and the HOS of the data points are also utilized for the purpose of clustering.**

I. Introduction

The problem of data clustering is quite extensively encountered in image processing. Important applications include image segmentation [1], pattern recognition [2], and image compression using vector quantization [3]. Recently, Rose *et al.* proposed a novel clustering method, in which the annealing process with its phase transitions leads to a natural hierarchical clustering [4]–[8]. One does not need to know the total number of clusters in advance. Rather, one has a natural way of deciding on the final number of clusters. Unlike traditional clustering methods [9]–[15], which are basically descent algorithms, the hierarchical deterministic annealing (HDA) clustering algorithm is insensitive to the choice of the initial configuration.

In this brief, we extend the HDA method to include two important additional features.

1) In [6], the Hessian corresponding to a cluster is computed by considering only the points which belong to that cluster, and hence, a split in the cluster is governed by only those points. In our method, the intercluster dependencies are also accounted for by considering the *entire* data distribution (and not just individual clusters) to compute the Hessian. As will be shown, the utility of considering intercluster dependency becomes significant when data points belonging to different clusters overlap.
2) The distortion function for which we analyze the phase transitions is quite general. In most clustering algorithms, a square error or weighted-square-error distortion measure is usually used. In effect, this amounts to assuming the underlying distribution of the data to be Gaussian [2]. But in practical applications, the distribution could be arbitrary. We propose a general distortion measure based on the higher order statistics (HOS) of the data.

Positive definiteness of the Hessian is used as a criterion for splitting the clusters. This criterion was chosen over the perturbation variant of the HDA method [8], because knowing to predict the next critical temperature allows acceleration of the annealing process between transitions, while being more careful during the transition. It may also be mentioned here that the idea of using the entire data distribution appears in [8], in the context of rate-distortion theory. While we express the condition for bifurcation as a general condition on the

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