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The Haar wavelet transform: its status and achievements

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Abstract

This paper is a brief survey of basic definitions of the Haar wavelet transform. Different generalizations of this transform are also presented. Sign version of the transform is shown. Efficient symbolic calculation of Haar spectrum is discussed. Some applications of Haar wavelet transform are also mentioned.

Keywords: Discrete transforms; Wavelet transforms; Haar functions; Haar transforms; Decision diagrams

1. Introduction

Haar functions have been used from 1910 when they were introduced by the Hungarian mathematician Alfred Haar [26]. The Haar transform is one of the earliest examples of what is known now as a compact, dyadic, orthonormal wavelet transform [7,33]. The Haar function, being an odd rectangular pulse pair, is the simplest and oldest orthonormal wavelet with compact support.

In the meantime, several definitions of the Haar functions and various generalizations have been published and used. They were intended to adapt this concept to some practical applications, as well as to extend its application to different classes of signals. Thanks to their useful features and possibility to provide a 'local' analysis of signals, the Haar functions appear very attractive in many applications as for example, image coding, edge extraction, and binary logic design

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[7,13–15,17,28,33,43]. The sample applies to many related concepts as the SHT [25], or the Watari transform [29,51] and the real multiple-valued Haar transform [53]. These transforms have been applied, for example, to spectral techniques for multiple-valued logic [29,53], etc. A quantized version of the Haar transform was recently developed [22,36].

This paper is an attempt to bring together these concepts published in the period of 90 years and by the authors in different parts of the World, and offer an unified basis for a further work in the area.

Due to space limitations, faced with the variety of definitions of the Haar and related functions, we first discuss definitions of the basic Haar functions. Then we present the most extensive and the most recent generalizations, since they involve as particular cases many other published and used definitions. However, references to these other results, or at least sources where they can be traced, are provided.

For applications of the Haar transform in logic design, efficient ways of calculating the Haar spectrum from reduced forms of Boolean functions are needed. Recently, such methods were introduced for calculation of the Haar spectrum from disjoint cubes [19,20], and different types of decision diagrams [15,16,27,42].

Finally, applications of the Haar transform in different fields are briefly discussed.

2. Haar functions

In Ref. [26], Alfred Haar has defined a complete orthogonal system of functions in $L_p[0,1]$, $p \in [1,\infty]$ taking values in the set $\{0, \sqrt{2^i}\}$, $i \in N_0$. This system reported now as the Haar functions, has property that each function continuous on $[0,1]$ can be represented by an uniformly convergent series in terms of elements of this system.

Definition 1. The Haar functions can be defined as follows:

$$\mathrm{har}(0,\theta) = 1, 0 \leqslant \theta \leqslant 1,$$

$$\mathrm{har}(1,\theta) = \begin{cases} 1, & 0 \leqslant \theta < 1/2, \\ -1, & 1/2 \leqslant \theta < 1, \end{cases}$$

$$\mathrm{har}(2,\theta) = \begin{cases} \sqrt{2}, & 0 \leqslant \theta < 1/4, \\ -\sqrt{2}, & 1/4 \leqslant \theta < 1/2, \\ 0, & 1/2 \leqslant \theta \leqslant 1. \end{cases}$$

$$\mathrm{har}(3,\theta) = \begin{cases} 0, & 0 \leqslant \theta < 1/2, \\ \sqrt{2}, & 1/2 \leqslant \theta < 3/4, \\ -\sqrt{2}, & 3/4 \leqslant \theta \leqslant 1, \end{cases}$$

$$\vdots$$

$$\mathrm{har}(2^p+n,\theta) = \begin{cases} \sqrt{2^p}, & n/2^p \leqslant \theta < (n+1/2)/2^p, \\ -\sqrt{2^p}, & (n+1/2)/2^p \leqslant \theta < (n+1)2^p, \\ 0, & 0 < \theta < \frac{n}{2^p} \text{ and } \frac{(n+1)}{2^p} < \theta < 1, \end{cases}$$

$$p = 1, \ldots; \quad n = 0, \ldots, 2^p - 1.$$

There are some other definitions of the Haar functions in the literature. However, they mutually differ with respect to the values of Haar functions at the points of discontinuity. In his original definition, Haar defined

$$\mathrm{har}(k, 0) = \lim_{\theta \to 0, \theta > 0} \mathrm{har}(k, \theta),$$

$$\mathrm{har}(k, 1) = \lim_{\theta \to 1, \theta < 0} \mathrm{har}(k, \theta),$$

and at the points of discontinuity within the interior (0,1) of the interval [0,1]

$$\mathrm{har}(k, \theta) = \tfrac{1}{2}(\mathrm{har}(k, \theta - 0) + \mathrm{har}(k, \theta + 0)).$$

Instead of that, some authors use

$$\mathrm{har}(k, \theta) = \mathrm{har}(k, \theta + 0),$$

where in the engineering practice it is usually assumed that the Haar function takes zero value at the points of the discontinuity.

Often, the two parametric notations for the Haar functions $\mathrm{har}(i, j, \theta)$ or $H_i^{(j)}(\theta)$ are used, where

$$H_0^{(0)}(\theta) = \mathrm{har}(0, \theta),$$

$$H_i^{(j+1)}(\theta) = \mathrm{har}(2^{i-1} + j, \theta), \quad i \in N_0, \ \ j = 1, \ldots, 2^i.$$

The parameter i is called the power of the Haar function and denotes a subset of Haar functions with the same number of zero crossings on the interval of the length $1/2^i$. Table 1 gives the grouping for the first 16 Haar functions with respect to i. Such ordering can be compared to the frequency ordering of trigonometric functions or the sequency ordering of Walsh functions [29,33]. Parameter j is denoted as the order of the Haar function. It determines the place of each function within the ith subset.

In two parametric notations, the following definition is possible.

Table 1
Sequency groupings of the first 16 Haar functions

Haar functions	Sequency
$\mathrm{har}(0, \theta)$	1
$\mathrm{har}(1, \theta)$	2
$\mathrm{har}(2, \theta)$, $\mathrm{har}(3, \theta)$	3
$\mathrm{har}(4, \theta)$, $\mathrm{har}(5, \theta)$	4
$\mathrm{har}(6, \theta)$, $\mathrm{har}(7, \theta)$	
$\mathrm{har}(8, \theta)$, $\mathrm{har}(9, \theta)$	5
$\mathrm{har}(10, \theta)$, $\mathrm{har}(11, \theta)$	
$\mathrm{har}(12, \theta)$, $\mathrm{har}(13, \theta)$	
$\mathrm{har}(14, \theta)$, $\mathrm{har}(15, \theta)$	

Definition 2. The Haar functions are defined by

$$\mathrm{har}(0,0,\theta) = 1, 0 \leqslant \theta \leqslant 1,$$

$$\mathrm{har}(i,j,\theta) = \begin{cases} \sqrt{2^i}, & \frac{j-1}{2^i} \leqslant \theta < \frac{j-1/2}{2^i}, \\ -\sqrt{2^i}, & \frac{j-1/2}{2^i} \leqslant \theta < \frac{j}{2^i}, \\ 0, & \text{otherwise} \end{cases} \quad i = 0,1,2,\ldots; \quad j = 1,\ldots,2^i.$$

2.1. Properties of Haar functions

From their definition, it is obvious that the Haar functions are orthogonal functions. Therefore,

$$\int_0^1 \mathrm{har}(m,\theta)\mathrm{har}(n,\theta)\,\mathrm{d}\theta = \begin{cases} 1, & n = m, \\ 0, & n \neq m. \end{cases}$$

The proof of completeness for the system of Haar functions is given by Haar himself [26].

Uljanov [49] have proved that if zero is assumed for the values for Haar functions in the points of the discontinuity, as in Definition 2, then uniform convergence of series in terms of the Haar functions is missing. In that way, the basic motive for introduction of the Haar functions in mathematical analysis, i.e., for uniform approximation in L_p [0,1], is not preserved. However, other properties of Haar functions, which make them applicable in engineering practice, and resulting advantages in numerical computations, make this pragmatic assumption justified and acceptable.

An outstanding property of the Haar functions is that except $\mathrm{har}(0,\theta)$, the ith Haar function can be obtained by the restriction of the $(i-1)$th function to the half of the interval where it is different from zero, by multiplication with $\sqrt{2}$ and by scaling over the interval [0,1]. That property caused renewed considerable interest in the Haar functions, since it closely relates them to the wavelet theory. In this setting, the first two Haar functions are called the global functions, while all the others are denoted as the local functions.

3. Discrete Haar functions

Discrete Haar functions can be defined as functions determined by sampling the Haar functions at 2^n points [3]. Alternatively, they can be considered independently as a particular set of functions in the Hilbert space of functions on the finite dyadic groups G_{2^n} of order 2^n defined by the analogy to the Haar functions [29]. Recall that the dyadic group of order 2^n is the direct product of n cyclic groups of order 2 $C_2 = (\{0,1\},\oplus)$, where $\oplus$ denotes the addition modulo 2.

Discrete Haar functions are conveniently represented as rows of an $(2^n \times 2^n)$, $n \in N$, matrix denoted as the Haar matrix. The Haar matrices are considered in the natural and sequency ordering which differ in the ordering of rows.

Definition 3. Discrete Haar functions of order n represented by $(2^n \times 2^n)$ matrix $H_s(n)$, in the sequency ordering are given by the following recurrence relation:

$$\mathbf{H}_s(n) = \begin{bmatrix} H_s(n-1) \otimes [\,1 \quad 1\,] \\ 2^{\frac{(n-2)}{2}}\mathbf{I}_2 \otimes \mathbf{I}_{2^{n-2}} \otimes [\,1 \quad -1\,] \end{bmatrix},$$

where

$$\mathbf{H}_s(1) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

and $\mathbf{I}_q$ is the identity matrix of order q.

The Haar matrix in the ordering corresponding to the natural or Hadamard ordering of the Walsh matrix [29,46], can be derived in the following way. The bit-reverse procedure is applied to the binary expressions of the indices of rows in the sequency ordered Haar matrix. Then, the indices are ordered in the increasing order of the values of their decimal equivalents within each subset. Such procedure is denoted as the zonal bit-reversal procedure.

Another definition of the Haar matrix $\mathbf{H}_s(n)$ can be given by using the generalized Kronecker product [46,52] defined as follows.

Definition 4. (*Generalized Kronecker product*). Denote by $\{\mathbf{A}\}$ the set of p unitary matrices $\mathbf{A}^i$, $i = 0, \ldots, p-1$ of order q and by $\{\mathbf{B}\}$ the set of q unitary matrices $\mathbf{B}^i$, $i = 0, \ldots, q-1$ of order p.

The generalized Kronecker product $\{\mathbf{A}\} \odot \{\mathbf{B}\}$ is the square $(pq \times pq)$ matrix $\mathbf{C} = [c_{ij}]$, where

$$c_{ij} = C_{up+w,u'p+w'} = A^w_{uu'} B^{u'}_{ww'},$$

where $A^w_{uu'}$ is the uu'th element of $\mathbf{A}^w \in \{\mathbf{A}\}$, $B^u_{ww'}$ is the ww'th element of $\mathbf{B}^{u'} \in \{\mathbf{B}\}$, where uu' and ww' are determined by the relations

$$i = up + w, \quad j = u'p + w'$$

$$u, u' = 0, \ldots, q-1; \quad w, w' = 0, \ldots, p-1.$$

With this definition, the sequency ordered Haar functions are defined by

$$\mathbf{H}_s(n) = \{\mathbf{H}_s(1), \sqrt{2}\mathbf{I}_2, \ldots, \sqrt{2}\mathbf{I}_2\} \odot \{\mathbf{H}_s(n-1), \mathbf{H}_s(n-1)\}.$$

Example 1. For $p = 4$, $q = 2$, $\mathbf{H}_s(3) = \{\mathbf{H}_s(1), \sqrt{2}\mathbf{I}_2, \sqrt{2}\mathbf{I}_2, \sqrt{2}\mathbf{I}_2\} \odot \{\mathbf{H}_s(2), \mathbf{H}_s(2)\}$.

Definition of the generalized Kronecker product is important, since for different choices of $\{\mathbf{A}\}$ and $\{\mathbf{B}\}$ and by using permutation matrices, a family of discrete transforms with the same fast calculation algorithms can be defined. It is denoted as the identical computation (IC) family of discrete transforms, whose particular examples are DFT, Walsh and Haar transform [46].

For example, the Walsh transform is defined for $\{\mathbf{A}\} = [\mathbf{F}_2, \ldots, \mathbf{F}_2]$. For the so-called slant transforms [25], SHT and slant Walsh (SWHT) transform, the recursive definition is as previously given except for a supplementary rotation of rows 1 and 2^{n-1} by the rotation matrix

$$\mathbf{F}_2(\theta_n) = \begin{bmatrix} \sin\theta_n & \cos\theta_n \\ \cos\theta_n & -\sin\theta_n \end{bmatrix},$$

where $\theta_n < \pi/2$ is given by $\cos\theta_n = 2^{n-1}/\sqrt{(2^{2n}-1)/3}$.

Similar as for the Walsh functions, the discrete Haar functions can be defined as the solutions of corresponding differential equations in terms of the Gibbs–Haar derivatives [44]. The same applies to the generalized Haar functions considered in Section 6.1. A further generalization of these results is given in Ref. [40].

Thanks to their recursive structure, the Haar functions in different orderings can be generated by using some suitably defined shift and copy procedures [32]. In this setting, they are particular members of the broad family of the so-called bridge functions [32,55]. Many other discrete functions derived by the combination of the Walsh and Haar functions, or as their suitable modifications, belong to the same family. They are used in definition of the corresponding transforms, see for example Refs. [23,33,45,53,55,56].

3.1. Non-normalized discrete Haar functions

For some applications, as spectral analysis of switching functions, it is more convenient to work with the non-normalized system of Haar functions, which in this case takes the values 0, 1, -1. The Haar functions are piecewise constant on subintervals of the length $1/2^i$. Thus, the interval [0,1] can be split into 2^m equal subintervals and the corresponding element from the set of natural numbers N can be assigned to each of them. With these assumptions, the non-normalized Haar functions are defined as follows [29].

Definition 5. Non-normalized Haar functions of order n are

$$H_0^{(0}(\theta) = 1,$$

$$H_j^{(q)}(\theta) = \begin{cases} 1, & \theta \in [(2q-2)2^y, (2q-1)2^y), \\ -1, & \theta \in [(2q-1)2^y, 2q2^y), \\ 0, & \text{at other points in } [0, 2^m). \end{cases}$$

$$y = m - i - 1, \quad i = 0, \ldots, m-1, \quad q = 1, \ldots, 2^i.$$

The non-normalized Haar functions are also considered in both natural and sequency ordering [29]. In applications of non-normalized Haar functions in switching theory and logic design, see for example Refs. [8,12–20,27–30,37,38,48,57], it is convenient to represent the discrete Haar functions in terms of switching variables as is shown in the following example.

Example 2. For $n = 3$, the relationships between discrete Haar functions and switching variables ordered in the descending value of indices can be expressed as follows:

$$\begin{array}{l} 1 \\ (1-2x_1) \\ (1-2x_2)\overline{x}_1 \\ (1-2x_2)x_1 \\ (1-2x_3)\overline{x}_2\overline{x}_1 \\ (1-2x_3)x_2\overline{x}_1 \\ (1-2x_3)x_2\overline{x}_1 \\ (1-2x_3)x_2x_1 \end{array} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \\ f(4) \\ f(5) \\ f(6) \\ f(7) \end{bmatrix} = \begin{bmatrix} S_f(0) \\ S_f(1) \\ S_f(2) \\ S_f(3) \\ S_f(4) \\ S_f(5) \\ S_f(6) \\ S_f(7) \end{bmatrix}.$$

In this case, we can consider the fixed-polarity Haar expressions, in the same way as the fixed-polarity Reed–Muller [11], arithmetic [18,21], and Walsh [12] expressions have been considered.

Example 3. For $n = 3$, the use of the negative literal for x_3, i.e., the permutation $\bar{x}_3$ and x_3, results into a permutation of columns in $\mathbf{H}(3)$ as $(0,1,2,3,4,5,6,7) \rightarrow (4,5,6,7,0,1,2,3)$.

A class of differently ordered discrete Haar functions with the order of columns of $\mathbf{H}(n)$ determined by a permutation of bits in the binary representation of the argument $x = (x_1,\dots,x_n)$, $x \in \{0,\dots 2^n - 1\}$, $x_i \in \{0,1\}$ in $\mathrm{har}(w,x)$ is considered in Ref. [30].

Example 4. For $n = 3$, the permutation $\sigma_0 = \begin{pmatrix} 123 \\ 321 \end{pmatrix}$ of bits in $x = (x_3,x_2,x_1)$ results in the permutation of columns of $\mathbf{H}(3)$ as $(0,1,2,3,4,5,6,7) \rightarrow (0,4,2,6,1,5,3,7)$.

In this class of discrete Haar functions, the total of $n!$ different orderings of Haar functions is considered, compared to 2^n orderings in fixed-polarity Haar expressions. These sets of orderings are disjoint, since, for example, in bits permutations, $\mathrm{har}(w,0)$ is always at the first position in the set of Haar functions.

It should be noted that for a given function f, each manipulation with arguments, as different polarity of literals, and permutation of bits in binary representations, corresponds to a permutation of elements in the vector of functional values for f. Since, the Haar transform is a wavelets like transform, permutation of functional values produces different number of non-zero coefficients in the Haar expressions for a given f. That property is exploited in Ref. [29] to minimize the cost of hardware in spectral synthesis by using Haar expressions. A method for minimization of the number of non-zero Haar coefficients for multiple-output switching functions by the total autocorrelation functions is proposed there. The method allows the total of $2^n!$ possible permutations of functional values and guarantees the maximum number of pairs of equal functional values for the input vectors $x = (x_1,\dots,x_n)$ which differ in the value of x_n.

4. Discrete Haar transform

Discrete Haar functions are kernel of the discrete Haar transform [3,29,30,33,45,46,50,56,57].

Definition 6. For f on G_{2^n} represented by the truth-vector $\mathbf{F}(n) = [f(0),\dots,f(2^n-1)]^{\mathrm{T}}$, the Haar spectrum $\mathbf{Y}_f(n) = [Y(0),\dots,Y(2^n-1)]^{\mathrm{T}}$ is given by:

$$\mathbf{Y}_f(n) = \mathbf{H}(n)\mathbf{F}(n),$$

$$\mathbf{F}(n) = \mathbf{H}(n)^{-1}\mathbf{Y}_f(n),$$

where $\mathbf{H}(n)$ is the Haar matrix in the corresponding ordering and $\mathbf{H}(n)^{-1}$ is its inverse over C. For non-normalized Haar matrix, the inverse $\mathbf{H}(n)^{-1}$ is equal to $\mathbf{H}^{\mathrm{T}}(n)$ when signs of the coefficients are only considered, where $\mathbf{H}^{\mathrm{T}}$ denotes the transpose of $\mathbf{H}$.

The normalized and non-normalized Haar transform differ in the appearance of the factor of normalization [29,46]. Definition of the same form applies to any other set of Haar, generalized

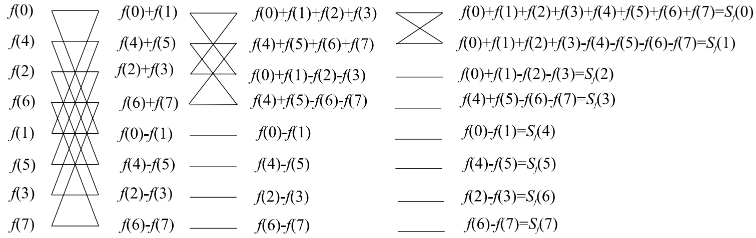

Fig. 1. FFT for the Haar transform for $n = 3$.

Haar and related functions if they are represented as the rows of a matrix of the corresponding order [33,46].

The Haar matrix $\mathbf{H}(n)$ can be factorized in a product of n sparse matrices permitting definition of FFT-like algorithms for computation of the Haar spectrum. These algorithms are denoted as the Fast Haar Transforms (FHTs), [30,33,46]. Different factorizations produce different FHTs with their properties adapted to some particular implementation technologies.

Example 5. Fig. 1 shows Cooley–Tukey FHT algorithm with bit-reverse reordering of input data for calculation of the Haar spectrum for $n = 3$. The Haar spectral coefficients are represented by symbols $S_f(i)$, $0 \leqslant i \leqslant 7$.

4.1. Decision diagrams based techniques for discrete Haar spectra

In spite of great theoretical interest in applications of the discrete Haar transform in switching theory and logic design [29,30,46,57], exponential complexity of FHT in terms of both space and time was a restrictive factor for wider practical applications of the Haar transform. Another reason is that up to recently, there has been no efficient method to calculate Haar spectrum directly from reduced representations of switching functions, such as cubes, and compact representations of large functions, as Decision Diagrams, and vice versa. A number of recent articles consider these important issues [15,16,19,20,27,41,42,47].

Paper [47] used digital circuit output probability and its interaction with Walsh spectral coefficients to calculate the Haar spectrum. In Refs. [19,20], the method to calculate Haar spectral coefficients from an array of disjoint cubes for systems of incompletely specified Boolean functions is presented.

Investigation of mutual relationships between ordered binary decision diagrams (OBDDs) and the Haar spectrum was done in Refs. [15,16]. A method to calculate Haar spectrum of switching functions from OBDDs has been presented there. The decomposition of the Haar spectrum in terms of cofactors of Boolean functions, has also been introduced in Refs. [15,16]. Based on the above decomposition, another method to synthesize OBDD directly from the Haar spectrum has been presented in Ref. [16]. A method to synthesize free binary decision diagram in quasi-optimal ordering form the Haar spectrum is shown in Ref. [8].

Calculation of the Haar spectrum for integer-valued functions defined on finite dyadic groups of large orders was solved in Ref. [42]. Advantages of this method are due to the peculiar properties of the Haar functions in natural ordering. However, the presented algorithm applies also to the calculation of the Haar spectrum in sequency ordering. Both normalized and non-normalized Haar spectra may be determined in both orderings after a simple rearrangement in the above algorithm.

As shown in Ref. [41], thanks to the recursive structure of the Haar matrix in Definition 3, the calculation of the Haar spectrum of a given function f can be performed through the multi-terminal binary decision diagrams (MTBDD) [9,10].

The method is derived from the following considerations. It should be noted that FHT consists of n steps, each step corresponding to a variable x_i in f. However, in the ith step, the processing is restricted to the subset of first 2^i output data from the $(i-1)$th step. The other data remain unprocessed and are simply sifted to the output of the algorithm. In these DD methods, it is assumed that a given function f is represented depending on its range by a MTBDD. Then, in each node of DD the basic FHT operation is performed over the co-factors f_0 and f_1 of f with respect to the variable assigned to the processed node. These co-factors are represented by subdiagrams rooted at the nodes pointed by the outgoing edges of the processed node. However, the mentioned property of the Haar transform, permits to restrict the calculation to the first values of the co-factors. This feature provides the efficiency of the implementation of FHT over DDs. The result of calculation at each node is stored in two fields assigned to each non-terminal node. The first field is used in further calculations, and the other field shows a particular Haar coefficient. The method will be illustrated by the following example.

Example 6. Fig. 2 shows calculation of the Haar transform over MTBDTs for $n = 3$. In this figure, $\mathbf{H}(1)$ denotes that at each node the calculations are performed as specified in the basic Haar matrix.

From this consideration, it follows that the Haar spectrum for a given function f can be calculated by using the following recurrence relations applied at the nodes and cross points in the MTBDD by starting from the constant nodes

$$\mathbf{Q}(N,k) = \mathbf{Q}(N_0,k-1)\hat{+}\mathbf{Q}(N_1,k-1)\Diamond\left(\sqrt{2^{n-k}}\,\hat{*}\,\mathbf{Q}(N_0,k-1)\hat{-}\mathbf{Q}(N_1,k-1)\right),$$

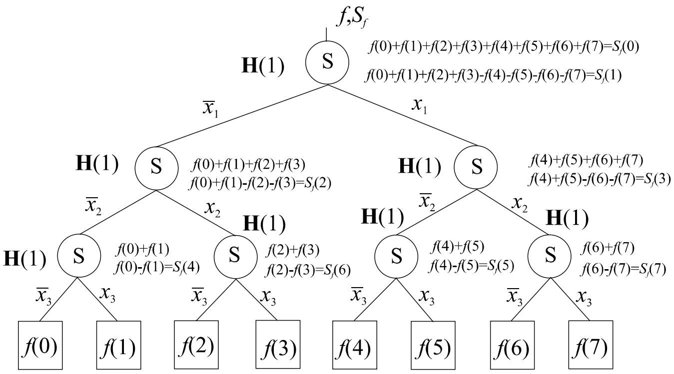

Fig. 2. Calculation of the Haar transform for $n = 3$ through MTBDT(f).

$$\mathbf{Q}(N,0) = v_N, \quad \text{if } N \text{ is the terminal node,}$$

where $\Diamond$ denotes concatenation of vectors, k is the node level, v_N is the value of the terminal node N, n is the number of variables and $\widehat{+}$, $\widehat{-}$ and $\widehat{*}$ are applied only for first elements in vectors and they denote addition, subtraction and multiplication, respectively.

Finally, the Haar spectrum is determined by:

$$\mathbf{Y}_f(n) = \mathbf{Q}(root, n).$$

The method will be illustrated by the following example taken from Ref. [41].

Example 7. Fig. 3 shows MTBDT for a function f on the finite dyadic group of order 2^3 given by the vector $\mathbf{F} = [1,1,2,0,2,0,2,0]^{\mathrm{T}}$.

The Haar spectrum of this function is calculated through the MTBDD as follows:

$$\mathbf{Q}(c,1) = ([2]\widehat{+}[0])\Diamond 2\widehat{*}([2]\widehat{-}[0]) = [2 \quad 4]^{\mathrm{T}}$$

$$\mathbf{Q}(c',1) = ([1]\widehat{+}[1])\Diamond 2\widehat{*}([1]\widehat{-}[1]) = [2 \quad 0]^{\mathrm{T}}$$

$$\mathbf{Q}(b,2) = \left(\begin{bmatrix}2\\0\end{bmatrix}\widehat{+}\begin{bmatrix}2\\4\end{bmatrix}\right)\Diamond\sqrt{2}\widehat{*}\left(\begin{bmatrix}2\\0\end{bmatrix}\widehat{-}\begin{bmatrix}2\\4\end{bmatrix}\right)$$
$$= [4,0,0,4]^{\mathrm{T}}$$

$$\mathbf{Q}(b',2) = \left(\begin{bmatrix}2\\4\end{bmatrix}\widehat{+}\begin{bmatrix}2\\4\end{bmatrix}\right)\Diamond\sqrt{2}\widehat{*}\left(\begin{bmatrix}2\\4\end{bmatrix}\widehat{-}\begin{bmatrix}2\\4\end{bmatrix}\right)$$
$$= [4,4,0,4]^{\mathrm{T}}$$

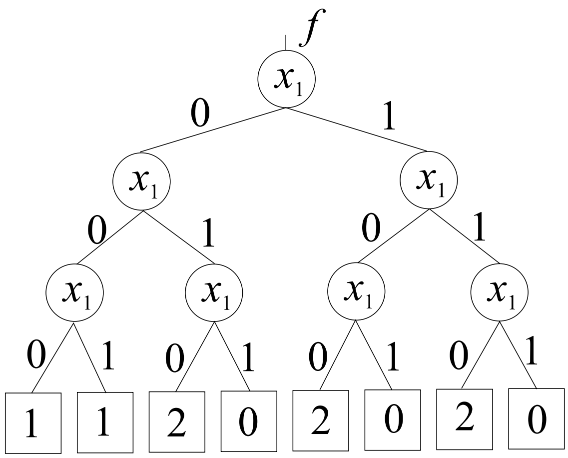

Fig. 3. MTBDT for f in Example 7.

$$\mathbf{Q}(a,3) = \left(\begin{bmatrix} 4 \\ 0 \\ 0 \\ 4 \end{bmatrix} \widehat{+} \begin{bmatrix} 4 \\ 4 \\ 0 \\ 4 \end{bmatrix} \right) \diamondsuit \left(\begin{bmatrix} 4 \\ 0 \\ 0 \\ 4 \end{bmatrix} \widehat{-} \begin{bmatrix} 4 \\ 4 \\ 0 \\ 4 \end{bmatrix} \right)$$
$$= [8,0,0,4,0,4,0,4]^{\mathrm{T}}$$

Another recent work on the calculation of non-normalized Haar spectra through decision diagrams is presented in Ref. [27]. This paper introduces a new data structure called the *Haar spectral diagram* (HSD) useful for representation of the Haar spectrum of Boolean functions. The natural ordering of Haar functions is used to represent the Haar transform matrix in terms of the Kronecker product providing a natural decision diagram based representation. The resulting graph is a point decomposition of the Haar spectrum using "0-element" edge values. For incompletely specified functions, the Haar spectrum represented as HSDs require no more nodes that the reduced OBDD for the same function, and for completely specified functions, the HSD is shown to be isomorphic to the reduced OBDD. The latter result is important, since it shows that by operating on the Haar spectrum no more storage is required that for the original function domain with all the information that the Haar spectral domain provides.

5. Sign Haar transform

Referring to the steps in fast calculation algorithm for the Haar transform, the sign Haar transform was introduced [22] by the analogy to the sign transform derived from the fast Walsh transform [2]. Besides calculation of forward and inverse sign Haar transform by using fast flow diagrams, such transforms can be calculated directly from recursive definitions that involves data and transform domain variables [22,24,36]. Many properties of sign Haar spectrum are similar to those of sign Walsh spectrum. The computational advantages of the Haar versus Walsh spectrum can be still extended to their corresponding sign transforms. It is therefore advantageous from the computational point of view to use sign Haar transform where sign Walsh transform have been used, i.e., for switching function decomposition and testing of logic circuits [2]. Besides applications in logic design, a new transform can be used when there is a need for a unique coding of binary/ternary vectors into the spectral domain of the same dimensions. One possible application is security coding in communication systems.

The following symbols are used in sign Haar transform definition and related discussions. Let

$$\mathbf{x}_n = [x_n, x_{n-1}, \dots, x_i, \dots, x_2, x_1], \quad \mathbf{w}_n = [w_n, w_{n-1}, \dots, w_i, \dots, w_2, w_1],$$

be n-tuples over the Galois field $GF(2)$. The symbol x_i stands for a data variable, and w_i for a transform domain variable; i is an integer and $1 \leqslant i \leqslant n$. Let

$$\mathbf{F} = [F_0, F_1, \dots, F_i, \dots, F_{2^n-2}, F_{2^n-1}]$$

be a ternary vector of symbols 0, +1, and −1, where the value of F_u $(0 \leqslant u \leqslant 2^{n-1})$ is given by $F(\mathbf{x}_n)$ when

$$\sum_{i=1}^{n} x_i 2^i = u.$$

Let

$$\mathbf{H}_F = [h_0, h_1, \dots, h_i, \dots, h_{2^n-2}, h_{2^n-1}],$$

be the vector corresponding to the sign Haar transform of $\mathbf{F}$. The value of h_u $(0 \leqslant u \leqslant 2^n - 1)$ is given by $\mathbf{H}_F(w)$ when

$$\sum_{i=1}^{n} w_i 2^i = u.$$

Let $\mathbf{O}_i$ represents the vector of i zeros, $1 \leqslant i \leqslant n$.

Let the symbols $\oplus_{\mathrm{c}}$, $\oplus_{\mathrm{d}}$, and $\wedge$ represent cyclic addition, dyadic addition, and bit-by-bit logic AND, respectively. When the above operations are applied to two vectors $\mathbf{A}_l$ and $\mathbf{B}_v$, $1 \leqslant l \leqslant v$, where l and v are two different integer numbers, they result in the vector $\mathbf{C}_v$ of the length v. Only l elements of $\mathbf{B}_v$ and all elements of $\mathbf{A}_l$ are manipulated, the remaining $(v - l)$ elements of the resulting vector $\mathbf{C}_v$ are not affected by the applied operation and are simply the same as elements of the vector $\mathbf{B}_v$ between positions v and $l + 1$.

We define

$$\operatorname{sign} z = \begin{cases} -1, & z < 0, \\ 0, & z = 0, \\ +1, & z > 0. \end{cases}$$

Definition 7. Forward sign Haar transform h is [22]

$$h(\mathbf{O}_n \oplus_{\mathrm{d}} w_1) = \operatorname{sign}\sum_{x_n=0}^{1}\left[\operatorname{sign}\sum_{x_{n-1}=0}^{1}\left[\cdots \operatorname{sign}\sum_{x_1=0}^{1}\{(-1)^{x_n w_1} f(\mathbf{x}_n)\}\cdots\right]\right].$$

For $1 \leqslant i \leqslant n$.

$$h(\mathbf{O}_n \oplus_{\mathrm{d}} \mathbf{w}_i \oplus_{\mathrm{d}} 2^i) = \operatorname{sign}\sum_{x_{n-i}=0}^{1}\left[\operatorname{sign}\sum_{x_{n-i-1}=0}^{1}\left[\dots \operatorname{sign}\sum_{x_1=0}^{1}\Big\{(-1)^{x_n-1} \times f\{[(\mathbf{O}_n \oplus_{\mathrm{d}} \mathbf{w}_i) \oplus_{\mathrm{c}} (n-i)] \oplus_{\mathrm{d}} \mathbf{x}_{n-i}\Big\}\cdots\right]\right].$$

Thus defined transform is an invertible transform.

Definition 8. The inverse sign Haar transform is defined by [22]

$$
\begin{aligned}
f(\mathbf{x}_n) = \operatorname{sign}\Bigg\{ (-1)^{x_1} h\{[(\mathbf{O}_1 \wedge \mathbf{x}_n) \oplus_{\mathrm{c}} 1] \oplus_{\mathrm{d}} 2^{n-1}\} + \operatorname{sign}\Bigg\{ (-1)^{x_2} h\{[(\mathbf{O}_2 \wedge \mathbf{x}_n) \oplus_{\mathrm{c}} 2] \oplus_{\mathrm{d}} 2^{n-2}\} \\
+ \cdots + \operatorname{sign}\Bigg\{ (-1)^{x_i} h\{[(\mathbf{O}_i \wedge \mathbf{x}_n) \oplus_{\mathrm{c}} i] \oplus_{\mathrm{d}} 2^{n-i}\} \\
+ \cdots + \operatorname{sign}\Bigg\{ (-1)^{x_{n-1}} h\{[(\mathbf{O}_{n-1} \wedge \mathbf{x}_n) \oplus_{\mathrm{c}} (n-1)] \oplus_{\mathrm{d}} 2\} \\
+ \operatorname{sign}\left[\sum_{w_1=0}^{1} (-1)^{x_n w_1} h(\mathbf{O}_n \oplus_{\mathrm{d}} w_1)\right] \Bigg\} \cdots \Bigg\}\Bigg\}\Bigg\}.
\end{aligned}
$$

Properties of the sign Haar spectra of Boolean functions were studied in Refs. [22,24]. An application of the sign Haar transform in ternary communication systems was considered in Ref. [36].

6. Generalizations

6.1. Haar functions on p-adic groups

The complex Haar functions defined in Ref. [35], can be considered as the extension of the non-normalized Haar functions to groups of order 4^n. Generalized Haar functions defined in Ref. [1], are a generalization of the Haar functions to any p-adic group. They are considered in different orderings related to the generalized translation operator derived from the associated group of permutations. In the sequency ordering, such generalized Haar functions can be also defined as follows.

Definition 9. Generalized Haar function $M_{r,s}^{(p,q)}(x)$ on p-adic groups are [29]:

$$M_{0,0}^{(p,1)}(x) = 1, \quad \forall x,$$

$$M_{r,s}^{(p,q)}(x) = \begin{cases} \mathrm{e}^{2\pi r i x^{(s)}/p}, & [(q-1)p^{m-s},\ qp^{m-s}), \\ 0, & \text{otherwise} \end{cases}$$

where $x^{(s)}$ is the sth coordinate in the p-adic expansion for x.

6.2. Zhang–Moraga Haar-type functions

Haar-type discrete functions and corresponding transforms are recently defined in R^N where R is a commutative ring with unity [54].

Assume that the Fourier transform with respect to the group characters of the underlying group G of R exists in R^N.

Definition 10. Kernel functions of a Haar-type transform are [54]:

$$HF^1_{0,0}(n) = 1$$

$$HF^t_{r_s,s}(n) = \begin{cases} w^{-q\frac{r_s n_s}{N_s}}, & \text{if } z, \\ 0, & \text{otherwise}, \end{cases}$$

$$z = (t-1)\prod_{i=s}^{m-1} N_i \leqslant n < t\prod_{i=s}^{m-1} N_i,$$

where w is the primitive qth root of unity in R and q is the exponent of G, and $r_s = 1, 2, \ldots N_g - 1$; $s = 0, 1, \ldots, m-1$, $t = 1, 2, \ldots, \prod_{i=s}^{m-1} N_i$, and $\prod_{i=s}^{m-1} N_i = 1$ if $s = 0$, while $n = \Sigma_{s=0}^{m-1} n_s \prod_{i=s+1}^{m-1} N_i$ and $\prod_{i=s+1}^{m-1} N_i = 1$ if $s = m-1$.

According to the equations in the above definition, we may get the Haar transform and Watari transform [51] in the complex number field, Haar number theoretic transform in the integer number ring modulo M, and Haar polynomial transform in the polynomial ring modulo $M(z)$ [54].

For p-adic additive group $(\{0, 1, \ldots, p-1\}^m, \oplus)$ of order 2^m, where $\oplus$ denotes addition modulo p, the kernel functions of the Haar-type transform may be expressed in the matrix form as follows:

$$\mathbf{HF} = \mathbf{T}_0\mathbf{T}_1 \cdots \mathbf{T}_{m-1},$$

where

$$\mathbf{T}_s = \text{diag}\{\mathbf{A}_{p^{s+1}}, \mathbf{I}_{p^{s+1}}, \ldots, \mathbf{I}_{p^{s+1}}\}, \quad s = 0, 1, \ldots, m-1,$$

where $\mathbf{I}_{p^{s+1}}$ stands for $(p^{s+1} \times p^{s+1})$ identity matrix

$$\mathbf{A}_{p^{s+1}} = \begin{bmatrix} \mathbf{I}_{p^s} \otimes \begin{bmatrix} r_{00} & r_{01} & \cdots & r_{0p-1} \end{bmatrix} \\ \mathbf{I}_{p^s} \otimes \begin{bmatrix} r_{10} & r_{11} & \cdots & r_{1_{p-1}} \end{bmatrix} \\ \mathbf{I}_{p^s} \otimes \begin{bmatrix} r_{p-10} & r_{p-11} & \cdots & r_{p-1p-1} \end{bmatrix} \end{bmatrix},$$

where

$$\begin{bmatrix} r_{00} & r_{01} & \cdots & r_{0p-1} \\ r_{10} & r_{11} & \cdots & r_{1p-1} \\ \vdots & & & \\ r_{p-10} & r_{p-11} & \cdots & r_{p-1p-1} \end{bmatrix} = \begin{bmatrix} w^0 & w^0 & \cdots & w^0 \\ w^0 & w^1 & \cdots & w^{p-1} \\ \vdots & & & \\ w^0 & w^{p-1} & \cdots & w^1 \end{bmatrix},$$

and w is a primitive root of unity in R and $\otimes$ stands for the Kronecker matrix product.

Now, let the matrix

$$\mathbf{Q} = \begin{bmatrix} r_{00} & r_{01} & \cdots & r_{0p-1} \\ r_{10} & r_{11} & \cdots & r_{1p-1} \\ \vdots & & & \\ r_{p-10} & r_{p-11} & \cdots & r_{p-1p-1} \end{bmatrix}$$

be another $(p \times p)$ orthogonal matrix in R. By using these formulas we can get some new 'Haar-type' transforms. For example, if the matrix $\mathbf{Q}$ is a $(p \times p)$ discrete cosine transform matrix [46]

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \cos\frac{\pi}{2p} & \cos\frac{3\pi}{2p} & \cdots & \cos\frac{(2p-1)\pi}{2p} \\ \cos\frac{2\pi}{2p} & \cos\frac{6\pi}{2p} & \cdots & \cos\frac{2(2p-1)\pi}{2p} \\ \vdots & & & \\ \cos\frac{(p-1)\pi}{2p} & \cos\frac{3(p-1)\pi}{2p} & \cdots & \cos\frac{(p-1)(2p-1)\pi}{2p} \end{bmatrix},$$

a new Haar-type transform called 'Haar-type cosine transform' it obtained in the complex number field. Its kernel functions may be expressed by

$$HC_{0,0}^{1}(n) = 1,$$

$$HC_{r,s}^{q}(n) = \begin{cases} \cos\frac{n_s+1/2}{\pi/p}, & \text{if } z, \\ 0, & \text{otherwise}, \end{cases}$$

$$z = (q-1)p^{m-s} \leqslant n < qp^{m-s},$$

where $n = 0, 1, \ldots, p^{m-1}$; $r = 1, 2, \ldots, p-1$; $s = 0, 1, \ldots, m-1$; $q = 1, 2, \ldots, p^s$, $n_s \in \{0, 1, \ldots, p-1\}$, and $n = \sum_{s=0}^{m-1} n_s p^{m-1-s}$.

The corresponding transform is defined as in Definition 10 by using these matrices as follows.

$$C_{0,0}^{1} = \frac{1}{N}\sum_{n=0}^{N-1} f(n),$$

$$C_{r,s}^{q} = p^{-m+s}\sum_{s=0}^{m-1} f(n)HC_{r,s}^{q}(n).$$

The following is its inverse transform

$$f(n) = C_{0,0}^{1} + 2\sum_{s=0}^{m-1}\sum_{r=1}^{p-1}\sum_{q=1}^{p^s} C_{r,s}^{q} HC_{r,s}^{q}(n).$$

If matrix $\mathbf{Q}$ is the $(p \times p)$ discrete Hartley transform matrix [46]

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \mathrm{cas}\left(\frac{2\pi}{p}\right) & \mathrm{cas}\left(\frac{4\pi}{p}\right) & \cdots & \mathrm{cas}\left(\frac{2(p-1)\pi}{p}\right) \\ \mathrm{cas}\left(\frac{4\pi}{p}\right) & \mathrm{cas}\left(\frac{8\pi}{p}\right) & \cdots & \mathrm{cas}\left(\frac{4(p-1)\pi}{p}\right) \\ \vdots & & & \\ \mathrm{cas}\left(\frac{2(p-1)\pi}{p}\right) & \mathrm{cas}\left(\frac{4(p-1)\pi}{2p}\right) & \cdots & \mathrm{cas}\left(\frac{2(p-1)(p-1)\pi}{p}\right) \end{bmatrix}$$

where $\mathrm{cas} = \cos\alpha + \sin\alpha$, we get a Haar-type Hartley transform in the complex number field. Its kernel functions are denoted by

$$RH^1_{0,0}(n) = 1,$$

$$RH^q_{r,s}(n) = \begin{cases} \mathrm{cas}\left(\frac{2\pi n_s}{p}\right), & \text{if } z, \\ 0, & \text{otherwise.} \end{cases}$$

$$z = (q-1)p^{m-s} \leqslant n < qp^{m-s}.$$

As is noted in Ref. [54], the above given definitions of the Haar-type matrix may be generalized to *mr*-adic additive groups to get *mr*-adic Haar-type transforms in various discrete function spaces.

7. Applications of the Haar transform

Due to its low computing requirements, the Haar transform has been mainly used for pattern recognition and image processing [7,33,52,57]. Hence, two dimensional signal and image processing is an area of efficient applications of Haar transforms due to their wavelet-like structure. In this area, it is usually reported that the simplest possible orthogonal wavelet system is generated from the Haar scaling function and wavelet. Moreover, wavelets are considered as a generalization of the Haar functions and transforms [3,6,50]. Such a transform is also well suited in communication technology for data coding, multiplexing and digital filtering [30,39,57]. For example, application of non-normalized Haar transform in a sequency division multiplex system is described in Ref. [55]. Bandwidth economy for multiplexed digital channels based on Haar transform is presented in [30]. For real time applications, hardware-based fast Haar chips have been developed [5,57]. In Ref. [4], different generalizations of Haar functions and transform are used in digital speech processing with applications in voice controlled computing devices and robotics. The control system based on Haar spectrum for military airplane is also discussed in Ref. [30]. The applications of Haar transform in control and communications are presented in Refs. [55,56]. In Ref. [34], different forms of Haar functions are used in approximate calculations of analytic functions. A brief discussion of various other applications, where the use of Haar and Walsh functions offers some advantages compared to the Fourier transform, is given in Ref. [57].

The advantages of computational and memory requirements of the Haar transform make it of a considerable interest to VLSI designers as well. For example, the authors of Refs. [37,38], presented a set of CAD tools to perform a switch-level fault detection and diagnosis of physical faults for practical MOS digital circuits using a reduced Haar spectrum analysis. In their system, the non-normalized Haar spectrum was used as a mean not only for diagnosing digital MOS ICs as a tool external to the circuit, but also as a possibility for a self-test strategy. The use of this set of CAD tools allowed the derivation of strategies for testing MOS circuits when memory states were encountered as a consequence of some type of faults. The advantage of using Haar functions instead of Walsh functions in CAD system based on spectra methods for some classes of Boolean functions was shown in Refs. [29,57].

For example, the analysis in Ref. [29] shows that the spectral complexity of conjunction and disjunction increases with the number of variables, exponentially for the Walsh functions and only linearly for the Haar functions. The circuit of the spectral multiplication logic module based on Haar functions was also developed [29,30]. It consists of a generator of basis functions, an adder, a multiplier, and the memory to store spectral coefficients. The module can be reprogrammed by changing dynamically its memory content. Such a behaviour of the module is useful in real-time adaptive control systems [30,57]. Karpovsky [29] noticed that the size of the memory block can be optimized only when the Harr basis is used. It is due to the fact that the number of non-vanishing Harr coefficients is reduced with input permutation of variables – the situation that does not exists for the Walsh basis. It should be noted that the realization of a permutation requires no special hardware [29]. Another advantage of the Harr spectrum in this and similar applications is the smallest number of required arithmetic operations.

In Ref. [48], a method for probabilistically determining the equivalence of two switching functions through Harr spectral coefficients has been developed. The method is reported as an alternative for equivalence checking of function that are difficult to represent completely and is based on BDDs and HDSs [27].

As is noted in Ref. [31], different generalization of spectral methods, including Haar functions, are intended to provide a unified theory for uniform consideration of different tasks in digital signal processing and related areas. Such theories are useful researches in both signal processing and applied mathematics. They bring new methods and tools for solving practical tasks to engineers, and trace and determine versatile and actual directions of research for mathematicians.

8. Closing remarks

This paper shows different generalizations and applications of Haar functions and transforms. Some recent developments and state-of-the art in Haar transforms are presented. The references are based not only on better known English language items, but also on lesser known entries from different Eastern European countries and China. The authors believe that this survey can be useful to researchers working in different disciplines where the Haar transform and closely related discrete wavelet transforms have been used.

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