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# Relationship between Haar and Reed-Muller spectral and functional domains

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**Abstract:** In this article, mutual relations between Haar and Reed-Muller spectral and functional domains are presented. The new relations apply to any size of the transform matrices in the form of layered vertical and horizontal Kronecker matrices. They allow the direct conversions between Haar and Reed-Muller functions and their corresponding spectra.

**Keywords:** Galois field (2), Haar, Reed-Muller, spectral coefficients

**Classification:** Science and engineering for electronics

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## 1 Introduction

Reed-Muller transform had been successfully applied in many areas such as signal processing, fault detection, and coding techniques, especially those



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concerned with group or block codes for error control [1]. One reason for the wide usage of Reed-Muller transform is because it has been found to be advantageous in terms of area, speed, and testability [2]. Another popular transform in spectral analysis, synthesis and testing and efficient representation of logic functions is Walsh transform [3]. An exact and non-exhaustive algorithm that generates optimal RM expansion for 3-variable binary functions directly from just few Walsh-Hadamard spectral coefficients had been developed in [4] while the case of one way conversion from Walsh to Reed-Muller spectra was discussed in [5].

Frequently, it is useful to apply more than one transform in a given task based on the local properties of a data function [3]. For example, if the logic function has many zeros in its truth vector, it is better to apply local transform that has non-zero entries in its own transformation matrix that almost overlap with the non-zero elements in the truth vector. In this case it is also of interest to investigate mutual relations between various local discrete transforms such as, for example, Haar and Reed-Muller transform. Both Haar and Reed-Muller transforms have been used in many applications of logic design [2, 3, 6, 7]. Therefore, it is not only interesting theoretically, but also practical to state their mutual relations. It should be noticed that Reed-Muller transform is performed over Galois Field (2) (GF(2)), while Haar transform is performed over integers.

In this article, mutual relations between Haar and Reed-Muller functional and spectral domains are presented in the form of matrix decomposition and as layered vertical and horizontal Kronecker product matrices for an arbitrary transform matrix size order. Due to the fact that Reed-Muller transform operations are in GF(2) some extra operations in this algebra are necessary.

## 2 Basic definitions

**Definition 1** The normalized Haar transform matrix of order $2^n$ is defined as [3, 6]

$$NH(n) = \begin{bmatrix} NH(n-1) & \otimes [1 \quad 1] \\ 2^{(n-1)/2} I_{2^{n-1}} & \otimes [1 \ -1] \end{bmatrix}, \tag{1}$$
$$NH(0) = [1], \quad n = 1, 2, 3, \ldots,$$

where $I_{2^{n-1}}$ is an identify matrix of $2^{n-1}$ while the symbol "$\otimes$" denotes Kronecker direct product [3, 6].

**Definition 2** The non-normalized Haar transform is obtained from replacing the nonzero entries of normalized Haar matrix with their arithmetic signs. The non-normalized Haar transform preserves all the properties of the normalized Haar transform. Its matrix of order $2^n$ is defined as [3, 6]

$$H(n) = \begin{bmatrix} H(n-1) \otimes [1 \quad 1] \\ I_{2^{n-1}} \quad \otimes [1 \ -1] \end{bmatrix}, \tag{2}$$
$$H(0) = [1], \quad n = 1, 2, 3, \ldots,$$

where $I_{2^{n-1}}$ is an identify matrix of $2^{n-1}$.


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**Definition 3** The matrix of order $2^n$ for Reed-Muller transform is defined as [2-5, 7]

$$RM(n) = \begin{bmatrix} RM(n-1) & 0 \\ RM(n-1) & RM(n-1) \end{bmatrix} \tag{3}$$
$$RM(0) = [1], \quad n = 1, 2, 3, \ldots.$$

$$\text{Also} \quad RM(n) = RM(1) \otimes RM(n-1) = \bigotimes_{i=1}^{n} RM(1) \tag{4}$$
$$\text{for} \quad n = 1, 2, 3 \ldots.$$

## 3 Relations between Haar and Reed-Muller functional and spectral domains

For an $n$-variable Boolean function $F(x_1, x_2, \ldots, x_n)$, Haar and Reed-Muller spectra (a column vector of dimension $2^n \times 1$) are given by $\vec{H} = H(n)\vec{F}$ and $\vec{R} = RM(n) \oplus \vec{F}$, respectively, where $\oplus$ denotes multiplication operation over Galois Field (2) (GF(2)). Hence, the calculation of Haar spectra $\vec{H}$ is performed in standard algebra while that of Reed-Muller spectra $\vec{R}$ is performed over GF(2). The following general relations are valid between the non-normalized Haar and Reed-Muller spectra: $\vec{H} = H(n)\left[RM(n) \oplus \vec{R}\right]$ and $\vec{R} = \left|RM^{-1}(n)H^{-1}(n)\vec{H}\right|$ where $H^{-1}(n)$ is inverse of the Haar transform matrix and $RM^{-1}(n)$ is inverse of Reed-Muller transform in standard and not GF(2) algebra, while the bracket $|\ |$ means taking modulo-2 of the absolute values of the result.

Let us now introduce the mutual relations between Haar and Reed-Muller transforms for a general case of an arbitrary $n$ using recursive definition for the transformation matrices. It should be noticed that presented in this article relations apply to conversions not only between Haar and Reed-Muller spectra but also between Haar and Reed-Muller functions. Therefore, in the following developments, the symbols for functions instead of spectra will be used. However, Reed-Muller and Haar functions can be freely replaced with Reed-Muller $\vec{R}$ and Haar $\vec{H}$ spectra, when needed. It is trivial to modify the presented equations for a case of normalized Haar functions as described in Definition 1 by adding normalizing factors. From Definition 2 presented earlier for non-normalized Haar transform matrix, it is obvious, that the first two rows of $H(n)$ are global basis functions $H_0(x)$ and $H_1(x)$, respectively. All subsequent rows are constituted by local basis functions $H_l^{(k)}(x)$ in an ascending order of $l$ and $k$. $l = 1, 2, ..$ is known as a degree of Haar function describing the number of zero crossings, and $k = 1, .., 2^l$ is an order of Haar function describing the position of the subset $l$ within a function. In Reed-Muller transformation matrix $RM(n)$, all but the last row are local basis functions and $RM_i$ denotes an $i$-th Reed-Muller function. The symbol $R_i^*$ denotes Reed-Muller spectrum obtained from the calculation using inverse Reed-Muller transformation matrix $RM^{-1}(n)$ in standard and not GF(2) algebra. The conversions for higher $n$ are shown for expressing non-normalized


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Haar functions by Reed-Muller functions and vice verse in the form of layered Kronecker product as follows.

$$\begin{bmatrix} H_0\\ H_1\\ H_2^{(1)}\\ H_2^{(2)}\\ H_3^{(1)}\\ H_3^{(2)}\\ H_3^{(3)}\\ H_3^{(4)}\\ \vdots\\ \vdots \end{bmatrix} = \left[\begin{array}{c} \begin{bmatrix}2&1\\0&-1\end{bmatrix} \otimes \left(\bigotimes_{i=1}^{n-1}[\,2\ \ 1\,]\right)\\ \hline \begin{bmatrix}1&0\\1&1\end{bmatrix} \otimes \left([\,0\ \ -1\,]\right) \otimes \left(\bigotimes_{i=1}^{n-2}[\,2\ \ 1\,]\right)\\ \hline \left(\bigotimes_{i=1}^{2}\begin{bmatrix}1&0\\1&1\end{bmatrix}\right) \otimes \left([\,0\ \ -1\,]\right) \otimes \left(\bigotimes_{i=1}^{n-3}[\,2\ \ 1\,]\right)\\ \hline \left(\bigotimes_{i=1}^{3}\begin{bmatrix}1&0\\1&1\end{bmatrix}\right) \otimes \left([\,0\ \ -1\,]\right) \otimes \left(\bigotimes_{i=1}^{n-4}[\,2\ \ 1\,]\right)\\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots\\ \hline \left(\bigotimes_{i=1}^{j-1}\begin{bmatrix}1&0\\1&1\end{bmatrix}\right) \otimes \left([\,0\ \ -1\,]\right) \otimes \left(\bigotimes_{i=1}^{n-j}[\,2\ \ 1\,]\right)\\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots\\ \hline \end{array}\right] \times \begin{bmatrix} RM_0^*\\ RM_1^*\\ RM_2^*\\ RM_{12}^*\\ RM_3^*\\ RM_{13}^*\\ RM_{23}^*\\ RM_{123}^*\\ \vdots\\ \vdots \end{bmatrix} \tag{5}$$

$$\begin{bmatrix} RM_0\\ RM_1\\ RM_2\\ RM_{12}\\ RM_3\\ RM_{13}\\ RM_{23}\\ RM_{123}\\ \vdots\\ \vdots \end{bmatrix} = \left| \frac{1}{2^n} \times \left[\begin{array}{c} \begin{bmatrix}1&1\\0&-2\end{bmatrix} \otimes \left(\bigotimes_{i=1}^{n-1}\begin{bmatrix}1\\0\end{bmatrix}\right) \vdots\\ \begin{bmatrix}1&0\\-1&1\end{bmatrix} \otimes \left(2\begin{bmatrix}1\\-2\end{bmatrix}\right) \otimes \left(\bigotimes_{i=1}^{n-2}\begin{bmatrix}1\\0\end{bmatrix}\right) \vdots\\ \left(\bigotimes_{i=1}^{2}\begin{bmatrix}1&0\\-1&1\end{bmatrix}\right) \otimes \left(4\begin{bmatrix}1\\-2\end{bmatrix}\right) \otimes \left(\bigotimes_{i=1}^{n-3}\begin{bmatrix}1\\0\end{bmatrix}\right) \vdots\\ \left(\bigotimes_{i=1}^{3}\begin{bmatrix}1&0\\-1&1\end{bmatrix}\right) \otimes \left(8\begin{bmatrix}1\\-2\end{bmatrix}\right) \otimes \left(\bigotimes_{i=1}^{n-4}\begin{bmatrix}1\\0\end{bmatrix}\right) \vdots\\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots\\ \left(\bigotimes_{i=1}^{j-1}\begin{bmatrix}1&0\\-1&1\end{bmatrix}\right) \otimes \left(2^{j-1}\begin{bmatrix}1\\-2\end{bmatrix}\right) \otimes \left(\bigotimes_{i=1}^{n-j}\begin{bmatrix}1\\0\end{bmatrix}\right) \vdots\\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \end{array}\right] \times \begin{bmatrix} H_0\\ H_1\\ H_2^{(1)}\\ H_2^{(2)}\\ H_3^{(1)}\\ H_3^{(2)}\\ H_3^{(3)}\\ H_3^{(4)}\\ \vdots\\ \vdots \end{bmatrix} \right| \tag{6}$$

In the above equations, there are two $2^n \times 2^n$ matrices, the vertical dotted lines denote the layered vertical Kronecker matrices, and the horizontal dashed lines denote the layered horizontal Kronecker matrices, respectively. A layered horizontal Kronecker matrix is defined as the horizontal sum of Kronecker matrices while a layered Kronecker vertical matrix is defined as the vertical sum of Kronecker matrices. It should be noticed, that the bracket $|\ |$ in Equation (6) is as defined previously. When Kronecker direct product


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of $j$ matrices is carried out for the above equations for $j \leq 0$, then the term $\bigotimes_{i=1}^{j}$ disappears from the above equations. The meaning of the symbols and the restriction on the term $\bigotimes_{i=1}^{j}$ is the same as above also for the following example.

**Example 1:** For $n = 3$, the above relations become:

$$
\begin{bmatrix} H_0 \\ H_1 \\ H_2^{(1)} \\ H_2^{(2)} \\ H_3^{(1)} \\ H_3^{(2)} \\ H_3^{(3)} \\ H_3^{(4)} \end{bmatrix}
=
\left[ \begin{array}{c}
\begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix} \otimes \left( \bigotimes_{i=1}^{3-1} \begin{bmatrix} 2 & 1 \end{bmatrix} \right) \\
\hdashline
\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \otimes \left( \begin{bmatrix} 0 & -1 \end{bmatrix} \right) \otimes \left( \bigotimes_{i=1}^{3-2} \begin{bmatrix} 2 & 1 \end{bmatrix} \right) \\
\hdashline
\left( \bigotimes_{i=1}^{2} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right) \otimes \left( \begin{bmatrix} 0 & -1 \end{bmatrix} \right) \otimes \left( \bigotimes_{i=1}^{3-3} \begin{bmatrix} 2 & 1 \end{bmatrix} \right)
\end{array} \right]
\times
\begin{bmatrix} RM_0^* \\ RM_1^* \\ RM_2^* \\ RM_{12}^* \\ RM_3^* \\ RM_{13}^* \\ RM_{23}^* \\ RM_{123}^* \end{bmatrix}
=
$$

$$
\left[ \begin{array}{c}
\begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} 2 & 1 \end{bmatrix} \otimes \begin{bmatrix} 2 & 1 \end{bmatrix} \\
\hdashline
\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} 2 & 1 \end{bmatrix} \\
\hdashline
\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & -1 \end{bmatrix}
\end{array} \right]
\times
\begin{bmatrix} RM_0^* \\ RM_1^* \\ RM_2^* \\ RM_{12}^* \\ RM_3^* \\ RM_{13}^* \\ RM_{23}^* \\ RM_{123}^* \end{bmatrix}
=
$$

$$
\begin{bmatrix}
8 & 4 & 4 & 2 & 4 & 2 & 2 & 1 \\
0 & 0 & 0 & 0 & -4 & -2 & -2 & -1 \\
0 & 0 & -2 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 & -1 & 0 & 0 & -2 & -1 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & -1 & 0 & -1 & 0 & -1 & 0 & -1
\end{bmatrix}
\times
\begin{bmatrix} RM_0^* \\ RM_1^* \\ RM_2^* \\ RM_{12}^* \\ RM_3^* \\ RM_{13}^* \\ RM_{23}^* \\ RM_{123}^* \end{bmatrix}
$$

$$
\begin{bmatrix} RM_0 \\ RM_1 \\ RM_2 \\ RM_{12} \\ RM_3 \\ RM_{13} \\ RM_{23} \\ RM_{123} \end{bmatrix}
=
\left| \frac{1}{2^3} \times
\left[ \begin{array}{c}
\begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix} \otimes \left( \bigotimes_{i=1}^{3-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \vdots \\
\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \otimes \left( 2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right) \otimes \left( \bigotimes_{i=1}^{3-2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \vdots \\
\left( \bigotimes_{i=1}^{2} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \right) \otimes \left( 4 \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right) \otimes \left( \bigotimes_{i=1}^{3-3} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)
\end{array} \right]
\times
\begin{bmatrix} H_0 \\ H_1 \\ H_2^{(1)} \\ H_2^{(2)} \\ H_3^{(1)} \\ H_3^{(2)} \\ H_3^{(3)} \\ H_3^{(4)} \end{bmatrix}
\right| =
$$


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$$
\left| \frac{1}{2^3} \times \begin{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix} & \otimes & \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \otimes & \begin{bmatrix} 1 \\ 0 \end{bmatrix} \vdots \\ \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} & \otimes & \left( 2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right) & \otimes & \begin{bmatrix} 1 \\ 0 \end{bmatrix} \vdots \\ \left( \bigotimes_{i=1}^{2} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \right) & \otimes & \left( 4 \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right) & & \\ & & & & \end{bmatrix} \times \begin{bmatrix} H_0 \\ H_1 \\ H_2^{(1)} \\ H_2^{(2)} \\ H_3^{(1)} \\ H_3^{(2)} \\ H_3^{(3)} \\ H_3^{(4)} \end{bmatrix} \right| =
$$

$$
\left| \frac{1}{2^3} \times \begin{bmatrix} 1 & 1 & 2 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -8 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & -4 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8 & -8 & 0 & 0 \\ 0 & -2 & -2 & 2 & -4 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 8 & 0 & -8 & 0 \\ 0 & 0 & 4 & -4 & 4 & -4 & -4 & 4 \\ 0 & 0 & 0 & 0 & -8 & 8 & 8 & -8 \end{bmatrix} \times \begin{bmatrix} H_0 \\ H_1 \\ H_2^{(1)} \\ H_2^{(2)} \\ H_3^{(1)} \\ H_3^{(2)} \\ H_3^{(3)} \\ H_3^{(4)} \end{bmatrix} \right|
$$

## 4 Conclusions

In this article, mutual relations between Haar and Reed-Muller functional and spectral domains for an arbitrary transform size have been shown. Reed-Muller transform can represent an arbitrary logic function by using EXOR and AND gates only [2]. The recent interest in applications of such expressions in logic synthesis is caused by their excellent properties for testability and the fact that many practical functions have a big content of strongly non-unate functions (e.g. parity, addition and multiplication) that are best realized by EXOR and AND expressions [2]. Also many multilevel circuits based on EXOR elements are more advantageous when area, speed and testability are of main concern [2]. Many other applications and achievements of Reed-Muller transform in finding logic symmetries [7] as well as in error correcting codes [1] are also well known. The recent applications of Haar transform in logic design are described in [6]. The mutual relations between these two transforms are introduced here through recursive equations in the form of layered vertical and horizontal Kronecker products. The presented relations allow transfer known results of spectral logic design in Reed-Muller domain to Haar domain and vice verse and compare efficiency of both approaches in different applications for large Boolean functions. Finally it should be also noticed that presented derivations based on layered matrices can be efficiently implemented in the form of operations on spectral decision diagrams [3] for both Reed-Muller and Haar transforms using software or as hardware operations using Look-Up Table cascades in a similar manner as Walsh function generator was implemented in [8].


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