FAST CLASSICAL AND QUANTUM FRACTIONAL HAAR WAVELET TRANSFORMS

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Abstract

The fractional Fourier transform (FRFT) is one-parametric generalization of the classical Fourier transform. FRFT was introduced in eighties and found a lot of applications in signal processing. The time and spectral domains are both the special cases of the fractional Fourier domain. They correspond to the 0th and 1st fractional Fourier domains, respectively. In this paper, we introduce the classical and quantum fractional Haar–Wavelet transforms and develop corresponding fast algorithms.

1. Introduction

The singular–value decomposition (SVD) and eigen–decomposition (ED) is a tool of both practical and theoretical importance in digital signal processing. The SVD an ED transforms are applicable to many image processing problems such as image coding and restoration, data compression, and power spectrum analysis. They are defined following way.

Let $\mathcal{M} = [M_k(i)]_{\omega,t=0}^{N-1}$ be an arbitrary discrete nonsingular $(N \times N)$–transform. We form two product $\mathcal{M}^t\mathcal{M}$ and $\mathcal{M}\mathcal{M}^t$, where "t" is the transpose symbol. Last matrices are symmetric and hence they have eigen–decompositions: $\mathcal{M}\mathcal{M}^t = \mathbf{V}\Lambda\mathbf{V}^+$, $\mathcal{M}^t\mathcal{M} = \mathbf{W}\Lambda\mathbf{W}^+$, where $\Lambda := \mathbf{diag}\{\lambda_0, \lambda_1, \ldots, \lambda_{N-1}\}$ and $+$ denote the Hermitian conjugate. Then, it is well known that we can express $\mathcal{M}$ as the singular value decomposition $\mathcal{M} = \mathbf{VDW}^+$, where

$$\overset{\mathbf{V}}{\overset{\|}{\begin{bmatrix} | & | & & | \\ \Phi_0(i) & \Phi_1(i) & \vdots & \Phi_{N-1}(i) \\ | & | & & | \\ \vee & \vee & & \vee \end{bmatrix}}} \quad \overset{\mathbf{W}}{\overset{\|}{\begin{bmatrix} | & | & & | \\ \Psi_0(i) & \Psi_1(i) & \vdots & \Psi_{N-1}(i) \\ | & | & & | \\ \vee & \vee & & \vee \end{bmatrix}}}$$

are matrices of eigen–vectors of $\mathcal{M}\mathcal{M}^t$ and $\mathcal{M}^t\mathcal{M}$ transforms, respectively, and $\mathbf{D} := \sqrt{\Lambda}$. If $\alpha_0, \ldots, \alpha_{N-1}$ are an arbitrary real numbers then

$$\mathcal{M}^{\alpha_0,\ldots,\alpha_{N-1}} := \mathbf{V}\left[\mathbf{D}(\lambda_0^{\alpha_0}, \ldots, \lambda_{N-1}^{\alpha_{N-1}})\right]\mathbf{W}^+ \quad (1)$$

is called *the multi–parametric fractional $\mathcal{M}$–transform.* If $\alpha_i = \alpha$, $\forall i = 0, 1, \ldots, N-1$ then this transform is called *fractional $\mathcal{M}$–transform.*

In 1937, Gondon wrote a paper called "Immersion of the Fourier transform in a continuous group of functional transformation" [2]. In 1961, Bargmann extended the fractional Fourier transform in his paper [1], in which he gave definition of the fractional Fourier transform, one based on Hermite polynomials as an integral transformation. If $H_n(\sqrt{2\pi}t)$ is a Hermite polynomial of order n then functions

$$\Psi_n(t) = \frac{2^{1/4}}{\sqrt{2^n n!}} H_n(\sqrt{2\pi}t)\exp(-\pi t^2) \quad (2)$$

for $n = 0, 1, 2, \ldots$ are eigen–functions of the Fourier transform

$$\mathcal{F}[\Psi_n(t)] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \Psi_n(t)e^{2\pi jt\tau}dt = \lambda_n\Psi_n(t),$$

with $\lambda_n = i^n$ being the eigen–value corresponding to the nth eigen–function. They form an orthogonal set of functions on the interval $(-\infty, \infty)$ with respect to weight function $e^{\pi t^2}$:

$$\langle\Psi_n(t) \mid \Psi_m(t)\rangle = \int_{-\infty}^{+\infty} e^{\pi t^2}\Psi_n(t)\Psi_m(t)dt =$$

$$= \int_{-\infty}^{\infty} e^{-\pi t^2}\frac{2^{\frac{1}{4}}}{2^n n!}H_n(\sqrt{2\pi}t)\frac{2^{\frac{1}{4}}}{2^m m!}H_m(\sqrt{2\pi}t)dt = \delta_{mn}.$$

According to Bargmann the fractional Fourier transform $\mathcal{F}^\alpha$ is defined through its the eigen–functions by

$$\mathcal{F}^\alpha = [\mathcal{F}^\alpha(\omega, t)] := \left[\sum_{n=0}^{\infty} \lambda_n^\alpha \Psi_n(\omega)\Psi_n(t)\right] =$$

$$= \frac{\sqrt{2}}{2^n n!}\left[e^{\pi(\omega^2+t^2)} \sum_{n=0}^{\infty} \lambda_n^\alpha H_n(\sqrt{2\pi}\omega) H_n(\sqrt{2\pi}t)\right] =$$

$$= \frac{e^{\frac{i\pi}{4}(\alpha - \mathrm{sgn} \sin\frac{\pi\alpha}{2})}}{\sqrt{\sin\frac{\pi\alpha}{2}}}\left[e^{j\pi \mathrm{ctg}(\frac{\pi\alpha}{2})(\omega^2 - 2\omega t \csc(\varphi) + t^2)}\right],$$

where $\mathcal{F}^\alpha(\omega, t)$ is the kernel of the fractional Fourier transform. Obviously, a functions $\Psi_n(t)$ are eigen–functions of the fractional Fourier transform $\mathcal{F}^\alpha[\Psi_n(t)] = \lambda_n^\alpha \Psi_n(t)$ corresponding to the nth eigen–values λ_n^α, $n = 0, 1, 2, \ldots$ Of course for $\alpha = 1$ $\mathcal{F}^1(\omega, t) = e^{j\omega t}$.

In 1980, Namias reinvented the fractional Fourier transform again in his paper [17]. This approach was extended by McBride and Kerr [16]. The fractional Fourier transform was restricted to pure mathematical purposes. Very few publications appeared. Then Mendlovic and Ozaktas introduced the fractional Fourier transform into the field of optics [18] in 1993. Afterwards, Lohmann [15] reinvented the fractional Fourier transform based on the Wigner–distribution function and opened the fractional Fourier transform to bulk–optics applications. In the series of papers [10],[19]–[22] authors developed the fast algorithms for a wide class of classical fractional transforms.

In this paper, we introduce the *classical and quantum fractional Haar–Wavelet transforms* and develop corresponding fast classical and quantum algorithms.

2. Classical Haar–Wavelet transforms

The Haar–Wavelet transform can be defined from the Haar functions and has the following factorization [9]:

$$\mathbf{HW}_{2^n} = \prod_{i=1}^{n}\left[\left(\mathcal{F}_2 \otimes I_{2^{n-i}}\right) \oplus I_{2^n - 2^{n-i+1}}\right]\Pi_{2^n}, \quad (3)$$

where $\mathcal{F}_2 = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ is the Walsh (2×2)–transform, where Π_{2^n} is the perfect shuffle permutation matrix [5]. Classical description of Π_{2^n} can be given by describing its effect on a given vector. If $\mathrm{v} = (v_0, v_1, \ldots, v_{2^n-2}, v_{2^n-1})$ is a 2^nD vector, then the vector $\mathrm{w} = \Pi_{2^n}\mathrm{v}$ is obtained by splitting v in half and the shuffling the top and bottom halves of the deck. Alternatively, a description of the matrix Π_{2^n}, in terms of its elements Π_{ij}, for $i, j = 0, 1, \ldots, 2^n - 1$, can be given as

$$\Pi_{ij} = \begin{cases} 1, & \text{if } j = i/2 \text{ and } i \text{ is even, or if } j = \frac{i-1}{2} + 2^{n-1} \\ & \text{and } i \text{ is odd,} \\ 0, & \text{otherwise.} \end{cases} \quad (4)$$

The short description of Π_{2^n} can be given by the left cyclic bit–shift of i–indexes $v_{(i_{n-1}, i_{n-2}, \ldots, i_1, i_0}$ $\Pi_{2^n}(i_{n-1}, i_{n-2}, \ldots, i_1, i_0) = (i_0, i_{n-1}, i_{n-2}, \ldots, i_1)$. Note, that $\Pi_{2^n}^t$ performs the right cyclic bit–shift operation, i.e. $\Pi_{2^n}^t(i_{n-1}, i_{n-2}, \ldots, i_1, i_0) = (i_{n-2}, \ldots, i_1, i_0, i_{n-1})$.

The perfect shuffle permutation matrix Π_{2^n} has the following factorization [5]:

$$\Pi_{2^n} = \prod_{i=2}^{n}(I_{2^{n-i}} \otimes \Pi_4 \otimes I_{2^{i-2}}),$$

where Π_4 is the "bit swap" operator, i.e., $\Pi_4(i_1, i_0) := (i_0, i_1)$.

There are two families of generalized Haar transforms [7]–[14]. The first family (discrete controlled) has the following form:

$$\mathbf{HW}_{2^n}^{(k_1,k_2,\ldots,k_n)} :=$$

$$= \prod_{i=1}^{n}\left[\left(I_{2^{k_i}} \otimes \mathcal{F}_2 \otimes I_{2^{n-i}}\right) \oplus I_{2^n - 2^{n-i+1-k_i}}\right]\Pi_{2^n}, \quad (5)$$

where the set of numbers $(k_1, k_2, \ldots, k_n)$ marks (and controls) the generalized Haar transforms, moreover, $0 \le k_1 \le 0$, $0 \le k_2 \le 1$, $\ldots$, $0 \le k_n \le n-1$. In particular, $H_{2^n}^{(0,0,\ldots,0)} = H_{2^n}$ is the standard Haar transform and $H_{2^n}^{(0,1,\ldots,n-1)} = W_{2^n}$ is the Walsh transform.

The second family (discrete and continuous controlled) contains the multi–parametric Haar–Wavelet transforms of the following form:

$$\mathbf{HW}_{(\varphi_1,\varphi_2,\ldots,\varphi_n)}^{(k_1,k_2,\ldots,k_n)} :=$$

$$= \prod_{i=1}^{n}\left[\left(I_{2^{k_i}} \otimes \mathbf{CS}_2(\varphi_i) \otimes I_{2^{n-i}}\right) \oplus I_{2^n - 2^{n-i+1-k_i}}\right]\Pi_{2^n}, \quad (6)$$

where $\mathbf{CS}_2(\varphi_i) := \begin{bmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{bmatrix}$. Obviously,

$\mathbf{HW}_{(\frac{\pi}{4},\frac{\pi}{4},\ldots,\frac{\pi}{4})}^{(k_1,k_2,\ldots,k_n)} = \mathbf{HW}_{2^n}^{(k_1,k_2,\ldots,k_n)}$ and $\mathbf{HW}_{(\frac{\pi}{4},\frac{\pi}{4},\ldots,\frac{\pi}{4})}^{(0,1,\ldots,n-1)} = \mathbf{HW}_{2^n}$.

For $\mathbf{CS}_2(\varphi)$ we have the following eigen–decomposition:

$$\mathbf{CS}_2(\varphi) = \begin{bmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{bmatrix} =$$

$$\begin{bmatrix} \cos\frac{\varphi}{2} & \sin\frac{\varphi}{2} \\ -\sin\frac{\varphi}{2} & \cos\frac{\varphi}{2} \end{bmatrix}\begin{bmatrix} 1 & \\ & -1 \end{bmatrix}\begin{bmatrix} \cos\frac{\varphi}{2} & -\sin\frac{\varphi}{2} \\ \sin\frac{\varphi}{2} & \cos\frac{\varphi}{2} \end{bmatrix} =$$

$$= \mathbf{Rot}_2\left(\frac{\varphi}{2}\right)\mathbf{D}_2(1,-1)\overline{\mathbf{Rot}}_2\left(\frac{\varphi}{2}\right), \quad (7)$$

where

$$\mathbf{Rot}_2\left(\frac{\varphi}{2}\right) = \begin{bmatrix} \cos\frac{\varphi}{2} & \sin\frac{\varphi}{2} \\ -\sin\frac{\varphi}{2} & \cos\frac{\varphi}{2} \end{bmatrix}, \ \mathbf{D}_2(1,-1) = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}.$$

In particular, $\mathcal{F}_2 = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} =$

$$= \begin{bmatrix} \cos\frac{\pi}{8} & \sin\frac{\pi}{8} \\ -\sin\frac{\pi}{8} & \cos\frac{\pi}{8} \end{bmatrix}\begin{bmatrix} 1 & \\ & -1 \end{bmatrix}\begin{bmatrix} \cos\frac{\pi}{8} & -\sin\frac{\pi}{8} \\ \sin\frac{\pi}{8} & \cos\frac{\pi}{8} \end{bmatrix} =$$

$$= \mathbf{Rot}_2\left(\frac{\pi}{8}\right)\mathbf{D}_2(1,-1)\overline{\mathbf{Rot}}_2\left(\frac{\pi}{8}\right). \quad (8)$$

where $\cos\frac{\pi}{8} = \frac{\sqrt{2+\sqrt{2}}}{2}$ and $\sin\frac{\pi}{8} = \frac{\sqrt{2-\sqrt{2}}}{2}$.

Obviously, $\mathbf{CS}_2^{(\alpha_1,\alpha_2)}(\varphi) =$

$$= \begin{bmatrix} \cos\frac{\varphi}{2} & \sin\frac{\varphi}{2} \\ -\sin\frac{\varphi}{2} & \cos\frac{\varphi}{2} \end{bmatrix}\begin{bmatrix} e^{2\pi j\alpha_1} & \\ & e^{\pi j\alpha_2} \end{bmatrix}\begin{bmatrix} \cos\frac{\varphi}{2} & -\sin\frac{\varphi}{2} \\ \sin\frac{\varphi}{2} & \cos\frac{\varphi}{2} \end{bmatrix} =$$

$$= \mathbf{Rot}_2\left(\frac{-\varphi}{2}\right)\mathbf{D}_2(e^{2\pi j\alpha_1}, e^{\pi j\alpha_2})\mathbf{Rot}_2\left(\frac{\varphi}{2}\right), \quad (9)$$

and $\mathcal{F}_2^{(\alpha_1,\alpha_2)} =$

$$= \begin{bmatrix} \cos\frac{\pi}{8} & \sin\frac{\pi}{8} \\ -\sin\frac{\pi}{8} & \cos\frac{\pi}{8} \end{bmatrix}\begin{bmatrix} e^{2\pi j\alpha_1} & \\ & e^{\pi j\alpha_2} \end{bmatrix}\begin{bmatrix} \cos\frac{\pi}{8} & -\sin\frac{\pi}{8} \\ \sin\frac{\pi}{8} & \cos\frac{\pi}{8} \end{bmatrix} =$$

$$= \mathbf{Rot}_2\left(\frac{\pi}{8}\right)\mathbf{D}_2(e^{2\pi j\alpha_1}, e^{\pi j\alpha_2})\overline{\mathbf{Rot}}_2\left(\frac{\pi}{8}\right) \quad (10)$$

are the fractional $\mathbf{CS}_2$ and Walsh-(2×2)–transforms.

3. Classical fractional Haar–Wavelet transforms

The substitution of (9) into (6) gives the SVD of Haar–Wavelet transforms

$$\mathbf{HW}_{(\varphi_1,\varphi_2,\ldots,\varphi_n)}^{(k_1,k_2,\ldots,k_n)} :=$$

$$= \prod_{i=1}^{n}\left[\left(I_{2^{k_i}} \otimes \mathbf{Rot}_2\left(\frac{\varphi_i}{2}\right) \otimes I_{2^{n-i}}\right) \oplus I_{2^n - 2^{n-i+1-k_i}}\right] \times$$

$$\times \prod_{i=1}^{n}\left[\left(I_{2^{k_i}} \otimes \mathbf{D}_2(1,-1) \otimes I_{2^{n-i}}\right) \oplus I_{2^n - 2^{n-i+1-k_i}}\right] \times$$

$$\times \prod_{i=1}^{n}\left[\left(I_{2^{k_i}} \otimes \overline{\mathbf{Rot}}_2\left(\frac{\varphi_i}{2}\right) \otimes I_{2^{n-i}}\right) \oplus I_{2^n - 2^{n-i+1-k_i}}\right]\Pi_{2^n} = \quad (11)$$

$$= [\mathbf{V}_{(\varphi_1,\ldots,\varphi_n)}^{(k_1,\ldots,k_n)}]\,\mathbf{D}_{2^n}\,[\mathbf{W}_{(\varphi_1,\ldots,\varphi_n)}^{(k_1,\ldots,k_n)}],$$

where $\mathbf{V}_{(\varphi_1,\ldots,\varphi_n)}^{(k_1,\ldots,k_n)} :=$

$$= \prod_{i=1}^{n}\left[\left(I_{2^{k_i}} \otimes \mathbf{Rot}_2\left(\frac{\varphi_i}{2}\right) \otimes I_{2^{n-i}}\right) \oplus I_{2^n - 2^{n-i+1-k_i}}\right]\Pi_{2^n}, \quad (12)$$

$$\mathbf{D}_{2^n} := \prod_{i=1}^{n}\left[\left(I_{2^{k_i}} \otimes \mathbf{D}_2(1,-1) \otimes I_{2^{n-i}}\right) \oplus I_{2^n - 2^{n-i+1-k_i}}\right], \quad (13)$$

$$\mathbf{W}_{(\varphi_1,\ldots,\varphi_n)}^{(k_1,\ldots,k_n)} :=$$

$$= \prod_{i=1}^{n}\left[\left(I_{2^{k_i}} \otimes \mathbf{Rot}_2\left(\frac{\varphi_i}{2}\right) \otimes I_{2^{n-i}}\right) \oplus I_{2^n - 2^{n-i+1-k_i}}\right]\Pi_{2^n}. \quad (14)$$

Now we can define two types of classical Haar–Wavelet transforms.

1. The $2n$–parametric fractional Haar–Wavelet transforms with a separable diagonal matrix

$$\mathbf{HW}_{(\varphi_1,\varphi_2,\ldots,\varphi_n)}^{(k_1,k_2,\ldots,k_n)}(\alpha_{1,1},\alpha_{1,2},\ldots,\alpha_{1,n},\alpha_{2,n}) :=$$

$$= \prod_{i=1}^{n}\left[\left(I_{2^{k_i}} \otimes \mathbf{Rot}_2\left(\frac{\varphi_i}{2}\right) \otimes I_{2^{n-i}}\right) \oplus I_{2^n - 2^{n-i+1-k_i}}\right] \times$$

$$\times \prod_{i=1}^{n}\left[\left(I_{2^{k_i}} \otimes \mathbf{D}_2(e^{2\pi j\alpha_{1,i}}, e^{\pi j\alpha_{2,i}}) \otimes I_{2^{n-i}}\right) \oplus I_{2^n - 2^{n-i+1-k_i}}\right] \times$$

$$\times \prod_{i=1}^{n}\left[\left(I_{2^{k_i}} \otimes \overline{\mathbf{Rot}}_2\left(\frac{\varphi_i}{2}\right) \otimes I_{2^{n-i}}\right) \oplus I_{2^n - 2^{n-i+1-k_i}}\right]\Pi_{2^n} :=$$

$$= \prod_{i=1}^{n}\left[\left(I_{2^{k_i}} \otimes \mathbf{CS}_2^{(\alpha_{1,i},\alpha_{2,i})}(\varphi_i) \otimes I_{2^{n-i}}\right) \oplus I_{2^n - 2^{n-i+1-k_i}}\right]\Pi_{2^n}. \quad (15)$$

In particular,

$$\mathbf{HW}_{2^n}^{(k_1,k_2,\ldots,k_n)}(\alpha_{1,1},\alpha_{1,2},\ldots,\alpha_{1,n},\alpha_{2,n}) =$$

$$= \prod_{i=1}^{n}\left[\left(I_{2^{k_i}} \otimes \mathbf{CS}_2^{(\alpha_{1,i},\alpha_{2,i})}\left(\frac{\pi}{8}\right) \otimes I_{2^{n-i}}\right) \oplus I_{2^n - 2^{n-i+1-k_i}}\right]\Pi_{2^n}. \quad (16)$$

and

$$\mathbf{HW}_{2^n}(\alpha_{1,1},\alpha_{1,2},\ldots,\alpha_{1,n},\alpha_{2,n}) =$$

$$= \prod_{i=1}^{n}\left[\mathbf{CS}_2^{(\alpha_{1,i},\alpha_{2,i})}\left(\frac{\pi}{8}\right) \otimes I_{2^{n-i}} \oplus I_{2^n - 2^{n-i+1}}\right]\Pi_{2^n} \quad (17)$$

are classical fractional Haar–Wavelet transform associated with $\mathbf{HW}_{2^n}^{(k_1,k_2,\ldots,k_n)}$ and $\mathbf{HW}_{2^n}$, respectively. We see that the fractional Haar–Wavelet transforms (15)–(17) have the Haar–like fast algorithms.

2. The 2^n–parametric fractional Walsh transform with a non-separable diagonal matrix

$$\mathbf{HW}^{(k_1,k_2,\ldots,k_n)}_{(\varphi_1,\varphi_2,\ldots,\varphi_n)}(\alpha_0,\alpha_1,\ldots,\alpha_{2^n-1}) :=$$

$$= \prod_{i=1}^{n}\left[\left(I_{2^{k_i}}\otimes \mathbf{Rot}_2\left(\frac{\varphi}{2}\right)\otimes I_{2^{n-i}}\right)\oplus I_{2^n-2^{n-i+1-k_i}}\right]\times$$

$$\times\left[\mathbf{D}_{2^n}(\alpha_0,\alpha_1,\ldots,\alpha_{2^n-1})\right]\times$$

$$\times\prod_{i=1}^{n}\left[\left(I_{2^{k_i}}\otimes\overline{\mathbf{Rot}}_2\left(\frac{\varphi}{2}\right)\otimes I_{2^{n-i}}\right)\oplus I_{2^n-2^{n-i+1-k_i}}\right]\Pi_{2^n}, \quad (18)$$

where $\mathbf{D}_{2^n}(\alpha_0,\alpha_1,\ldots,\alpha_{2^n-1})$ is a diagonal $(2^n\times 2^n)$–matrix.

4 Quantum fractional Haar–Wavelet transform

All operations in quantum computation are realized by means of transformations on the QU–BIT's contained in a quantum register. The possible transformations a quantum computer can carry out are the elements of unitary group $\mathcal{U}(\mathbf{C}^{2^n})$. A *quantum logic gate* is an elementary quantum computing device which performs a fixed unitary transformation on selected QU–BIT's in a fixed period of time. A transformation gate takes an input quantum state and produces a modified output quantum state. The gates have the same number of inputs as outputs, and a gate of n inputs carries a unitary transformation of the group $\mathcal{U}(\mathbf{C}^{2^n})$, i.e., a generalized rotation in the Hilbert space $\mathbf{C}^{2^n}$. To study the complexity of performing unitary transformations on QU–2^nREG, we introduce two types of quantum logic gates [4]–[6],[23]:

• *Local unitary operations* on k–th QU–BIT are matrices of the form $U_{2^n}^{(k)} := I_{2^{k-1}}\otimes U_2\otimes I_{2^{n-k}}$, where U_2 is an element of the unitary group $\mathcal{U}(\mathbf{C}^2)$ of (2×2)–matrices. For these operations we have

$$\left[I_{2^{i-1}}\otimes U_2\otimes I_{2^{n-i}}\right]|q_1\rangle\otimes\ldots\otimes|q_i\rangle\otimes\ldots\otimes|q_n\rangle =$$

$$= |q_1\rangle\otimes\ldots\otimes\left[U_2|q_i\rangle\right]\otimes|\ldots\otimes|q_n\rangle. \quad (19)$$

• For any unitary $[2^{n-k}\times 2^{n-k}]$–transformation $U_{2^{n-k}}$ we define the n–BIT transformation $U_{2^n}^{\overline{1,k}}$ by

$$U_{2^n}^{\overline{1,k}} := I_{2^n-2^{n-k}}\oplus U_{2^{n-k}}. \quad (20)$$

This operator is called the $(\overline{1,k})$*–controlled* U_{2^n}*–operator*, where $U_{2^n}^{\overline{1,k}}$ acts as identity transforms in the subspace $\mathbf{C}^{2^n-2^{n-k}}$ and as $U_{2^{n-k}}$ in the second subspace $\mathbf{C}^{2^k}$, if $q_1,q_2,\ldots,q_k = 1$ and $q_{k+1},\ldots,q_n \neq 0$. Here $C^{2^n} = \mathbf{C}^{2^n-2^{n-k}}\oplus\mathbf{C}^{2^k}$. In particular, if $U_{2^{n-k}}$ is the tensor product of $n-k$ (2×2)–matrices $U_{2^{n-k}} := U_{2,k+1}\otimes U_{2,k+2}\otimes\cdots\otimes U_{2,n}$, then

$$\left[U_{2,k+1}\otimes\cdots\otimes U_{2,n}\right]_{2^n}^{\overline{1,k}}\times$$

$$\times\left[|q_1\rangle\otimes\ldots\otimes|q_k\rangle\otimes|q_{k+1}\rangle\otimes\ldots\otimes|q_n\rangle\right] =$$

$$|q_1\rangle\otimes\ldots\otimes|q_k\rangle\otimes\left[U_{2,k+1}^{\overline{1,k}}|q_{k+1}\rangle\right]\otimes\ldots\otimes\left[U_{2,n}^{\overline{1,k}}|q_n\rangle\right]. \quad (21)$$

If $U_{2^{n-k}} := U_{2^{n-k}}^{(j)} = I_{2^{j-1}}\otimes U_2\otimes I_{2^{n-k-j}}$, then

$$\left[I_{2^{j-1}}\otimes U_2\otimes I_{2^{n-k-j}}\right]_{2^n}^{\overline{1,k}}\left[|q_1\rangle\otimes\ldots\otimes|q_{k+j}\rangle\otimes\ldots\otimes|q_n\rangle\right] =$$

$$= |q_1\rangle\otimes\ldots\otimes\left[U_2^{\overline{1,k}}|q_{k+j}\rangle\right]\otimes\ldots\otimes|q_n\rangle. \quad (22)$$

• For any diagonal unitary (2×2)–transformation $\mathbf{D}_2^{(t_1,\ldots,t_{n-1},0)}$ we define the $(2^n\times 2^n)$–transformation by

$$\mathbf{D}_{2^n}^{(t_1,\ldots,t_{n-1},0)} = \mathbf{D}_{2^n}\left(e^{2j\pi\alpha(t_1,\ldots,t_{n-1},0)}, e^{j\pi\alpha(t_1,\ldots,t_{n-1},1)}\right) =$$

$$= I_{(t_1,\ldots,t_{n-1},0)}\oplus\mathbf{D}_2^{(t_1,\ldots,t_{n-1},0)}\oplus I_{(\bar{t}_1,\ldots,\bar{t}_{n-1},0)} \quad (23)$$

This operator is called the $(t_1,\ldots,t_{n-1},0)$*–controlled operator*. Obviously,

$$\mathcal{QD}_{2^n}^{(t_1,\ldots,t_{n-1},0)} =$$

$$= \prod_{t_1=0}^{1}\cdots\prod_{t_{n-1}=0}^{1}\left[I_{(t_1,\ldots,t_{n-1},0)}\oplus\mathbf{D}_{2^n}^{(t_1,\ldots,t_{n-1},0)}\oplus I_{(\bar{t}_1,\ldots,\bar{t}_{n1-1},0)}\right] \quad (24)$$

We shall use a standard graphical notation for quantum circuits. [4]–[6],[23] In this notation the tensor structure of the Hilbert space $\mathbf{C}^{2^n} = \mathbf{C}^2\otimes\mathbf{C}^2\otimes\ldots\otimes\mathbf{C}^2$ is reflected by drawing n parallel lines (=quantum wires) each of which represents one tensor component $\mathbf{C}^2$. A box sitting just on one wire denotes a local transformation $U_{2^n}^{(i)}$ whereas the $(\overline{1,k})$–controlled $U_{2^{n-k}}^{\overline{1,k}}$–gate occupies all n wires: k for the control and $n-k$ for the transformation (see Fig. 1).

The *quantum network* (gate array)is a quantum computing device consisting of quantum logic gates whose computational steps are synchronised in time. The quantum network is the natural quantum generalization of the acyclic *combinatorial logic circuits* studied in conventional computational complexity theory. The output of some of the gates are connected by wires to the input of others and they interconnected without fanout or feedback by *quantum wires*. A *quantum computer* will be viewed here as a quantum network (or a family of quantum networks). *Quantum computation* is defined as unitary evolution of the network which takes its initial state "input" into some final state "output".

In order to realize quantum fast fractional Haar–Wavelet transforms, we introduce

- *input "time" quantum register*

$$\text{QU–}2^n\text{REG}(|\mathbf{t}\rangle) := \boxed{|t_1\rangle}\boxed{|t_2\rangle}\boxed{\cdots}\boxed{|t_n\rangle}$$

- *output "frequency" quantum register*

$$\text{QU–}2^n\text{REG}(|\boldsymbol{\omega}\rangle) := \boxed{|\omega_1\rangle}\boxed{|\omega_2\rangle}\boxed{\cdots}\boxed{|\omega_n\rangle}$$

According to (19), (20) we can introduce quantum counterparts of transforms (12), (14), (15) and (18)

$$\mathcal{QHW}^1_{2^n} = \prod_{i=1}^{n}\left[\mathbf{CS}_2^{\alpha_{1,i},\alpha_{2,i}}\right]_{2^n}^{\overline{(i+1,n)}},$$

$$\mathcal{QV}_{2^n} = \prod_{i=1}^{n}\left[\mathbf{Rot}_2\left(\frac{\varphi}{2}\right)\right]^{\overline{(i+1,n)}},$$

$$\mathcal{QW}_{2^n} = \prod_{i=1}^{n}\left[\overline{\mathbf{Rot}}_2\left(\frac{\varphi}{2}\right)\right]^{\overline{(i+1,n)}}\Pi_{2^n},$$

$$\mathcal{QHW}^2_{2^n} = \left[\mathcal{QV}_{2^n}\right]\left[\mathcal{QD}_{2^n}^{(t_1,\ldots,t_{n-1},0)}\right]\left[\mathcal{QW}_{2^n}\right].$$

In the language of quantum circuits, these transforms are presented in Fig. 2 and Fig. 3, respectively.

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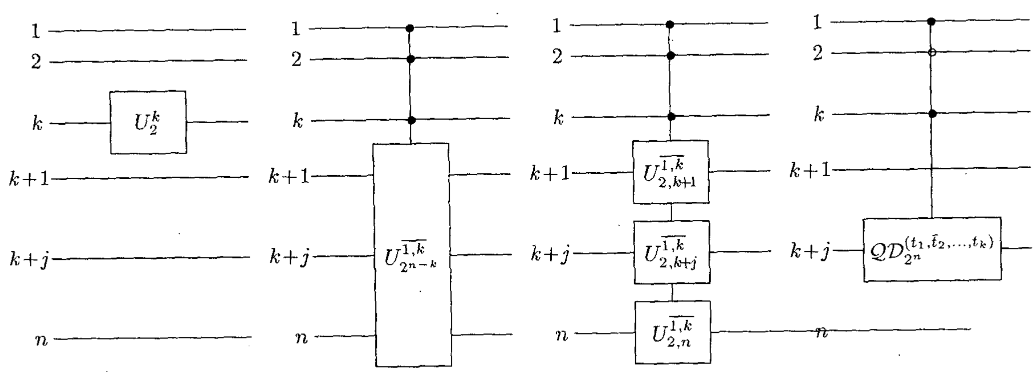

Figure 1. Quantum gates for a) $U_{2^n}^{(k)}$, b) $(\overline{1,k})$–controlled $U_{2^{n-k}}^{\overline{1,k}}$–operator, c) $\left[U_{2,k+1}^{\overline{1,k}} \otimes \cdots \otimes U_{2,n}^{\overline{1,k}}\right]$ and d) $\mathcal{QD}_{2^n}^{(t_1,\bar{t}_2,\ldots,t_k)}$, respectively

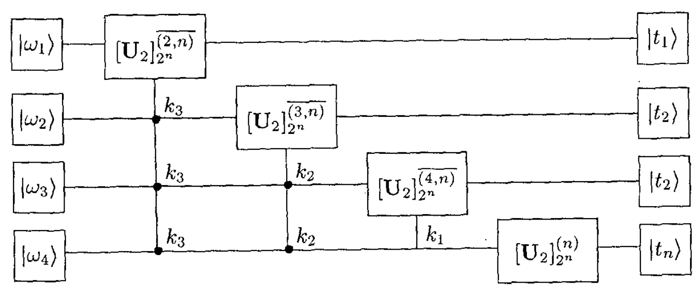

Figure 3. Quantum fast 2^n–parametric Haar–Wavelet transform $\mathcal{QHW}_{2^n}^{1}$, if $[\mathbf{U}_2]_{2^n}^{\overline{(2,n)}} = \left[\mathbf{CS}_2^{\alpha_{1,i},\alpha_{2,i}}\right]_{2^n}^{\overline{(i+1,n)}}$ or left $\mathcal{QV}_{2^n}$ (right $\mathcal{QW}_{2^n}$) eigen–transforms, if $[\mathbf{U}_2]_{2^n}^{\overline{(2,n)}} = \left[\mathbf{Rot}_2\left(\frac{\varphi}{2}\right)\right]^{\overline{(i+1,n)}}$, for $n = 4$

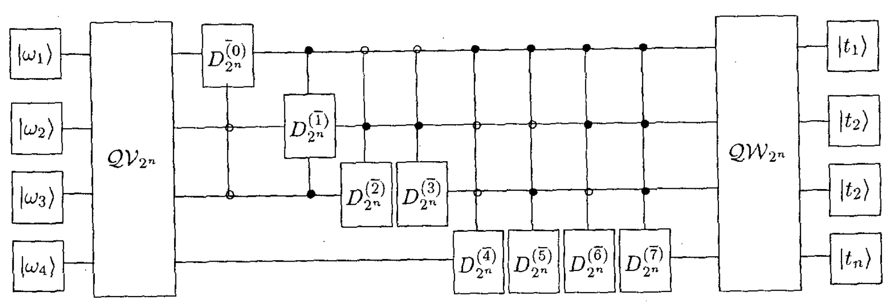

Figure 3. Quantum fast 2^n–parametric Haar–Wavelet transform $\mathcal{QHW}_{2^n}^{(\alpha_0,\alpha_1,\ldots,\alpha_{2^n-1})}$, $n = 4$