Fourier Transform

Decomposition of signals in terms of sinusoids

\[ A e^{i\omega} = A \cos(\omega x) + iA \sin(\omega x) \]

Using complex exponential as an input

\[ y[n] = e^{i\omega n} h[n] = \sum_{k=-\infty}^{\infty} h[k] e^{i\omega (n-k)} = e^{i\omega n} \sum_{k=-\infty}^{\infty} h[k] e^{-ik} \]

\[ y[n] = e^{i\omega n} H(\omega) \]

Given complex exponential as input, output is again complex exponential scaled by \( H(\omega) \)

\( H(\omega) \) is the frequency response of linear-time invariant systems

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Fourier Series

Central theme – approximate a function with given a family of basis functions

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Fourier Series and Transform

Decomposition of signals in terms of sinusoids - Fourier series

\[ f[x] = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} c_n \cos(nx) + c_n \sin(nx) \]

\[ f[x] = \frac{1}{\pi} \sum_{n=-\infty}^{\infty} a_n \cos(nx) + b_n \sin(nx) \]

Fourier series and transform

\[ f[x] = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} c_n e^{ijn} \quad \text{and} \quad c_n = \frac{1}{2\pi} \sum_{-\infty}^{\infty} f[x] e^{-ijn} \]

Alternative form

\[ f[x] = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} F[n] e^{ijn} \quad \text{and} \quad F[n] = \frac{1}{2\pi} \sum_{-\infty}^{\infty} f[x] e^{-ijn} \]

Central theme – approximate a function with given a family of basis functions

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Fourier Transform

• Connecting back to LTI
• Frequency response of a linear time invariant system is the Fourier transform of the unit-impulse response
• i.e. LTI are uniquely characterized by their impulse response and equivalently by their frequency response

\[ F[f] = \sum_{n=-\infty}^{\infty} f[n] e^{-i\omega n} \quad \text{and} \quad \langle F[f], F[g] \rangle = \text{conv}^{-1}(F[f], F[g]) \]

\[ \langle F[f], F[g] \rangle = \int F[f](\omega) F[g](\omega) \, d\omega \]

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Fourier transform as representation

• The spatial information is lost - spectral information is gained
• Can we achieve simultaneous localization in space (time) and frequency? -> only within some bounds
• Windowed Fourier Transform - balance between the two

Filtering - Fourier Transform

• Examples FT phase and frequency information
  \[ g[x] = f[x] \ast h[x] \quad G[\omega] = F[\omega] \ast H[\omega] \]
• Low pass filtering
• High pass filtering
• Band pass filtering
• Differentiation

Fourier transform and filtering


Fourier transform and denoising

Low Pass filtering

High pass filtering

Band-pass filtering

FFT by example

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Scale Space Representations

- Gaussian Pyramids
  enable to extract different structures in the image - since different structures are more apparent at different scales
  (useful for search over scale - detection, spatial search, feature tracking)
  \[ I(x, y, \sigma) = I_0(x, y) \ast G(x, y, \sigma), \]
  (Witkin'83, Koenderink, van Doorn'86)
  Family of functions - solutions to diffusion eq.
  \[ \frac{\partial I}{\partial \sigma} = \frac{\partial I}{\partial x^2} + \frac{\partial I}{\partial y^2} = \nabla^2 I \]
  Courtesy T. Lindenberg, KTH

- Causality (features at coarse level - have cause at fine level)
- Homogeneity and Isotropy (blurring is space-invariant)

Anisotropic Diffusion

Problem with traditional scale space models
Gaussian is symmetric - smoothes over edges
Does not preserve localization of edges
Idea - edge preserving smoothing

\[ \frac{\partial I}{\partial \sigma} = \frac{\partial I}{\partial x^2} + \frac{\partial I}{\partial y^2} = \nabla^2 I \quad I(x, y, 0) \]

With spatially varying term (Perona & Malik '90)

\[ \frac{\partial I}{\partial \sigma} = \nabla \cdot \left( \epsilon(x, y, \sigma) \nabla I \right) = \epsilon(x, y, \sigma) \nabla^2 I + (\nabla \epsilon(x, y, \sigma) \cdot \nabla I) \]

If we knew where are the edges - we can
Create a mask \( \epsilon(x, y, \sigma) \) (assuming that \( \epsilon(x, y, \sigma) \) is independent of \( \sigma \))
We get noise free image and smooth where there are no edges

Courtesy T. Lindenberg, KTH
Anisotropic Diffusion - Example

Images courtesy P. Kovesi (www.csee.unwa.au/~pk)
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