Fourier Transform

Reference

– Chapter 2.2 – 2.5, Carlson, Communication Systems

Using the Fourier series, a signal over a finite interval can be represented in terms of a complex exponential series. If the function is periodic, this representation can be extended over the entire interval $(-\infty, \infty)$.

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_o t} \qquad F_n = \frac{1}{T}\int_0^T f(t) e^{-jn\omega_o t} dt$$

Fourier Transform

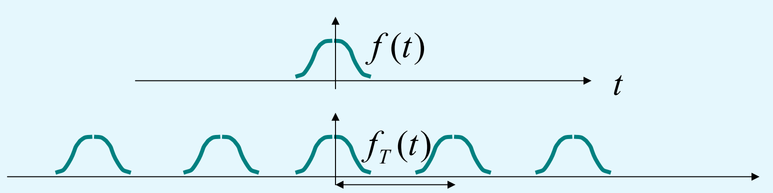
On the other hand, Fourier transform provides the link between the time-domain and frequency domain descriptions of a signal. Fourier transform can be used for both periodic and non-periodic signals.

$$f(t) = \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega \qquad F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

Example

The the last example of the previous lecture shows that if the period (T) of a periodic signal increases, the fundamental frequency ($\omega_o=2\pi/T$) becomes smaller and the frequency spectrum becomes more dense while the amplitude of each frequency component decreases. The shape of the spectrum, however, remains unchanged with varying T. Now, we will consider a signal with period approaching infinity.

Suppose we are given a non-periodic signal $f(t)$. In order to applying Fourier series to the signal $f(t)$, we construct a new periodic signal $f_T(t)$ with period T.



The original signal can be obtained back again by letting the period $T \to \infty$, that is,

$$f(t) = \lim_{T\to\infty} f_T(t)$$

The periodic function $f_T(t)$ can be represented by an exponential Fourier series.

$$f_T(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_o t} \quad \text{where}$$

$$F_n = \frac{1}{T}\int_{-T/2}^{T/2} f_T(t) e^{-jn\omega_o t} dt \quad \text{and} \quad \omega_o = 2\pi / T$$

As the magnitude of the Fourier coefficients go to zero when the period is increased, we define

$$\omega_n \equiv n\omega_o \qquad \text{and} \qquad F(\omega_n) \equiv T \cdot F_n$$

The Fourier series pair become

$$f_T(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T} F(\omega_n) e^{j\omega_n t} \quad \text{where} \quad F(\omega_n) = \int_{-T/2}^{T/2} f_T(t) e^{-j\omega_n t} dt$$

The spacing between adjacent lines ($\Delta\omega$) in the line spectrum of $f_T(t)$ is

$$\Delta\omega = 2\pi / T$$

Therefore, we have

$$f_T(t) = \sum_{n=-\infty}^{\infty} F(\omega_n) e^{j\omega_n t} \frac{\Delta\omega}{2\pi} \qquad (1)$$

Now as T becomes very large, $\Delta\omega$ becomes smaller and the spectrum becomes denser. In the limit, the discrete lines in the spectrum of $f_T(t)$ merge and the frequency spectrum becomes continuous.

Mathematically, the infinite sum (1) becomes an integral

$$\lim_{T\to\infty} f_T(t) = \lim_{T\to\infty} \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} F(\omega_n)e^{j\omega_n t}\Delta\omega$$

$$\Rightarrow \boxed{f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega}$$

Inverse Fourier transform of *F(ω)*

Similarly,

$$F_n = \frac{1}{T}\int_{-T/2}^{T/2} f_T(t)e^{-jn\omega_o t}\,dt$$

$$\Rightarrow F(\omega_n) = \int_{-T/2}^{T/2} f_T(t)e^{-j\omega_n t}\,dt \qquad \because \omega_n \equiv n\omega_o \text{ and } F(\omega_n) \equiv TF_n$$

$$\Rightarrow \lim_{T\to\infty} F(\omega_n) = \lim_{T\to\infty}\int_{-T/2}^{T/2} f_T(t)e^{-jn\omega_o t}\,dt$$

$$\Rightarrow \boxed{F(\omega) = \int_{-T/2}^{T/2} f_T(t)e^{-jn\omega_o t}\,dt}$$

Fourier transform of *f(t)*

Operators are often used to denote the transform pair.

$\Im\{f(t)\}$ Fourier transform of *f(t)*

$\Im^{-1}\{F(\omega)\}$ Inverse Fourier transform of $F(\omega)$

$$f(t) = \Im^{-1}\left[\Im\{f(t)\}\right]$$

Singularity functions

- This is a particular class of functions which are useful in signal analysis.
- They are mathematical idealization and, strictly speaking, do not occur in physical systems.
- Good approximation to certain limiting condition in physical systems. For example, a very narrow pulse.

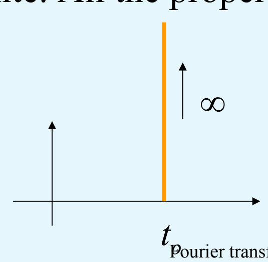
Singularity functions

Impulse function

- This function has the property exhibited by the following integral:

$$\int_a^b f(t)\delta(t-t_o)dt = \begin{cases} f(t_o) & a < t_o < b \\ 0 & \text{elsewhere} \end{cases} \qquad (2)$$

for any *f(t)* continuous at $t = t_o$, t_o is finite. All the properties can be derived from this definition.



Singularity functions

Properties of the impulse function

Amplitude

All values of $\delta(t)$ for $t \neq t_o$ are zero.

The amplitude at the point $t = t_o$ is undefined.

Area (strength)

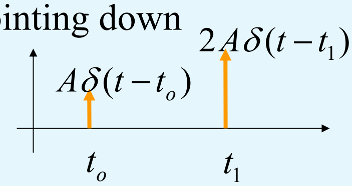
If $f(t) = 1$, (2) becomes

$$\int_a^b \delta(t - t_o)\,dt = 1 \qquad a < t_o < b$$

Therefore $\delta(t)$ has unit area. Similarly, $A\delta(t)$ has an area of A units.

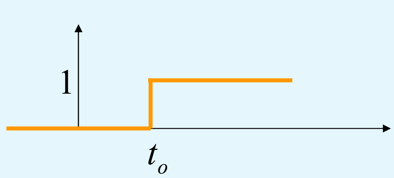
Graphic representation

To display the impulse function at $t = t_o$, an arrow is used to avoid to display the amplitude. The area of the impulse is designated by a quantity in parentheses beside the arrow or by the height of the arrow. An arrow pointing down indicates negative area.



Relation to the unit step function

The unit step function is defined by



$$u(t-t_0) = \begin{cases} 1 & t > t_o \\ 0 & t < t_o \end{cases}$$

Using (2) and letting $a = -\infty, b = t$, and $f(t) = 1$

$$\int_a^b f(t)\delta(t-t_o)dt$$

$$\Rightarrow \int_{-\infty}^{t} \delta(\tau - t_o)d\tau = \begin{cases} 1 & t > t_o \\ 0 & t < t_o \end{cases} \Rightarrow \boxed{\int_{-\infty}^{t} \delta(\tau - t_o)d\tau = u(t-t_o)}$$

Therefore the integral of the unit impulse function is the unit step function. The converse can also be shown by differentiating both sides of the above equation.

$$\int_{-\infty}^{t} \delta(\tau - t_o)d\tau = u(t-t_o)$$

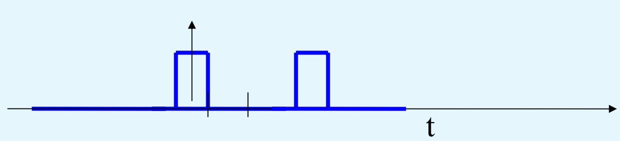
$$\Rightarrow \frac{d}{dt}\int_{-\infty}^{t} \delta(\tau - t_o)d\tau = \frac{d}{dt}u(t-t_o)$$

$$\Rightarrow \boxed{\delta(t-t_o) = \frac{d}{dt}u(t-t_o)}$$

Spectral density function $F(\omega)$

$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$ represents $f(t)$ as a continuous sum of exponential functions with frequencies lying in the interval $(-\infty, \infty)$. The relative amplitude of the components at any frequency ω is proportional to $F(\omega)$. $F(\omega)$ is called the spectral density function of $f(t)$.

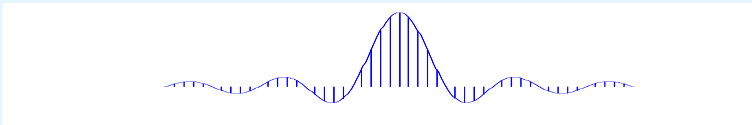
– Each point on the $F(\omega)$ curve contributes nothing to the representation of $f(t)$; it is the area that contributes.



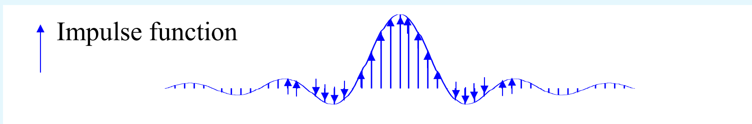
Spectral density function $F(\omega)$

Example

- Consider a rectangular pulse train
- The line spectrum of the Fourier series of the signal is



The spectral density of the signal is



Existence of the Fourier transform

We may ignore the question of the existence of the Fourier transform of a time function when it is an accurately specified description of a physically realizable signal. In other words, physical realizability is a sufficient condition for the existence of a Fourier transform.

Parseval's theorem for energy signals

Using the Parseval's theorem, we can find the energy of a signal in either the time domain or the frequency domain.

$$\boxed{E = \int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega}$$

Example

Energy contained in the frequency band $\omega_1 < \omega < \omega_2$ of a real-valued signal is

$$\frac{1}{2\pi}\left[\int_{-\omega_2}^{-\omega_1} |F(\omega)|^2 d\omega + \int_{\omega_1}^{\omega_2} |F(\omega)|^2 d\omega\right]$$

$$= \frac{1}{\pi}\int_{\omega_1}^{\omega_2} |F(\omega)|^2 d\omega$$

Fourier transforms of some signals

Impulse function δ(*t*)

The Fourier transform of a unit impulse δ(*t*) is

$$\Im\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = e^{j0} = 1 \qquad \because \int_{-\infty}^{\infty} \delta(t) f(t) dt = f(0)$$

It shows that an impulse function has a uniform spectral density over the entire frequency spectrum. In practice, a narrow pulse in time domain has a very wide bandwidth in frequency domain.

Example

If we increase the transmission rate of digital signal, a wider frequency bandwidth is needed.

Fourier transforms of some signals

Complex exponential function

The Fourier transform of a complex exponential function is

$$\Im\left\{e^{\pm j\omega_o t}\right\} = \int_{-\infty}^{\infty} e^{\pm j\omega_o t} e^{-j\omega t} dt = ?$$

On the other hand, the inverse Fourier transform of $\delta(\omega \pm \omega_o)$ is

$$\Im^{-1}\left\{\delta(\omega \pm \omega_o)\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega \pm \omega_o) e^{j\omega t} dt = \frac{1}{2\pi} e^{\pm j\omega_o t}$$

$$\therefore \Im\left\{e^{\pm j\omega_o t}\right\} = \Im\left\{\Im^{-1}\left\{\delta(\omega \pm \omega_o)\right\}\right\} = 2\pi\delta(\omega \pm \omega_o)$$

Sinusoidal signals

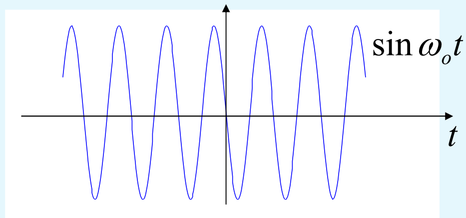
The sinusoidal signals can be written as

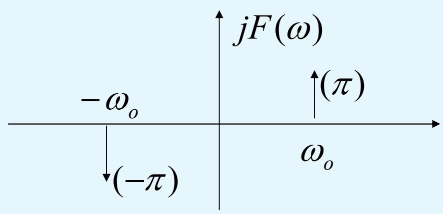
$$\cos x = \frac{e^{jx} + e^{-jx}}{2} \qquad \sin x = \frac{e^{jx} - e^{-jx}}{2j}$$

The Fourier transform of these signals are

$$\Im\left\{\cos\omega_o t\right\} = \Im\left\{\frac{e^{j\omega_o t} + e^{-j\omega_o t}}{2}\right\} = \pi\delta(\omega - \omega_o) + \pi\delta(\omega + \omega_o)$$

$$\Im\left\{\cos\omega_o t\right\} = \Im\left\{\frac{e^{j\omega_o t} - e^{-j\omega_o t}}{2j}\right\} = \frac{\pi}{j}\delta(\omega - \omega_o) - \frac{\pi}{j}\delta(\omega + \omega_o)$$





Periodic signal

A periodic signal *f*(*t*) can be represented by its exponential Fourier series.

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_o t} \qquad \text{where } \omega_o = 2\pi / T$$

The Fourier transform is

$$\begin{aligned} \Im\{f(t)\} &= \Im\left\{\sum_{n=-\infty}^{\infty} F_n e^{jn\omega_o t}\right\} \\ &= \sum_{n=-\infty}^{\infty} F_n \Im\{e^{jn\omega_o t}\} \\ &= 2\pi \sum_{n=-\infty}^{\infty} F_n \delta(\omega - n\omega_o) \end{aligned}$$

Thus the spectral density of a periodic signal consists of a set of impulses located at the harmonic frequencies of the signal. The area of each impulse is 2π times the values of its corresponding coefficient in the exponential Fourier series.

Some properties of the Fourier transform

Linearity (superposition)

The Fourier transform is a linear operation based on the properties of integration and therefore superposition applies.

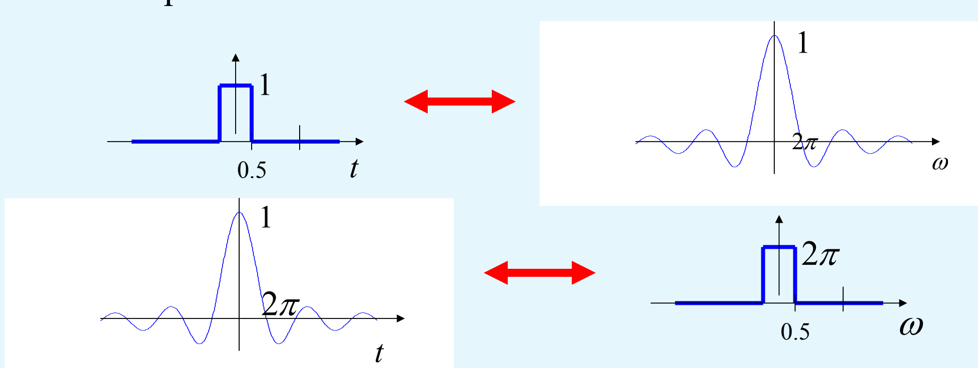
$$\Im\{af(t)+bg(t)\}=aF(\omega)+bG(\omega)$$

Some properties of the Fourier transform

Duality

If $\Im\{f(t)\}=F(\omega)$, then $\Im\{F(t)\}=2\pi F(-\omega)$

Example

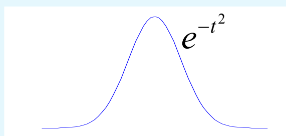


Coordinate scaling

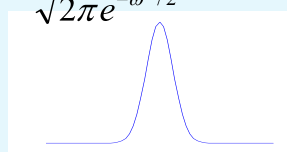
The expansion or compression of a time waveform affects the spectral density of the waveform. For a real-valued scaling constant α and any signal *f*(*t*)

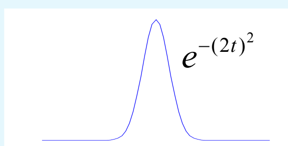
$$\Im\{f(\alpha t)\} = \frac{1}{|\alpha|} F\left(\frac{\omega}{\alpha}\right)$$

Example

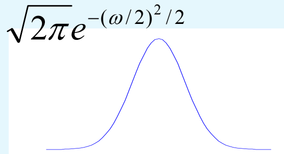


$\sqrt{2\pi}e^{-\omega^2/2}$





$\sqrt{2\pi}e^{-(\omega/2)^2/2}$



Time shifting (delay)

$$\Im\{f(t-t_o)\} = F(\omega)e^{-j\omega t_o}$$

If a signal is delayed in time by t_o, its magnitude spectral density remains unchanged and a negative phase $-\omega t_o$ is added to each frequency component.

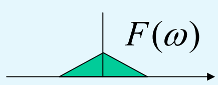
Frequency shifting (Modulation)

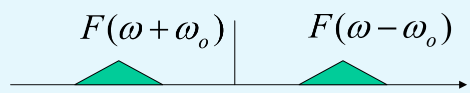
$$\Im\left\{ f(t)e^{j\omega_o t} \right\} = F(\omega - \omega_o)$$

Therefore multiplying a time function by $e^{j\omega_o t}$ causes its spectral density to be translated in frequency by ω_o.

Example

$$\Im\left\{ f(t)\cos\omega_o t \right\} = \frac{1}{2}\left[F(\omega + \omega_o) + F(\omega - \omega_o) \right]$$





Differentiation

$$\Im\left\{ \frac{d}{dt} f(t) \right\} = j\omega F(\omega)$$

Time differentiation enhances the high frequency components of a signal.

Integration

$$\Im\left\{ \int_{-\infty}^{t} f(\tau)\,d\tau \right\} = \frac{1}{j\omega}F(\omega) + \pi F(0)\delta(\omega) \qquad \text{where} \quad F(0) = \int_{-\infty}^{\infty} f(t)\,dt$$

Integration in time suppresses the high-frequency components of a signal.