Synthesis of Ternary Quantum Logic Circuits by Decomposition

Faisal Shah Khan† and Marek M. Perkowski*

† Department of Mathematic and Statistics
Portland State University, PO Box 751, Portland, Oregon 97207-0751;
* Department of Electrical and Computer Engineering
Portland State University, PO Box 751, Portland, Oregon 97207-0751

ABSTRACT

Recent research in multi-valued logic for quantum computing has shown practical advantages for scaling up a quantum computer. [1,12] Multivalued quantum systems have also been used in the framework of quantum cryptography, [4] and the concept of a qudit cluster state has been proposed by generalizing the qubit cluster state. [5] An evolutionary algorithm based synthesizer for ternary quantum circuits has recently been presented, [2] as well as a synthesis method based on matrix factorization [3].In this paper, a recursive synthesis method for ternary quantum circuits based on the Cosine-Sine unitary matrix decomposition is presented.

Keywords: Quantum Logic Synthesis, Cosine-Sine Decomposition, Ternary Quantum Logic

PACS numbers: 03.67.Lx, 03.65.Fd 03.65.Ud

1. INTRODUCTION

A collection of controlled three state quantum systems (qutrits) perturbed by a classical force can result in the state of one system controlling the evolution of a second; this is called a quantum circuit. Qutrits replace classical ternary bits as information units in ternary quantum computing. They are represented as a unit vector in state space, which is a complex three dimensional vector space, $\mathcal{H}_3$. In the computational basis, the basis vectors (or basis states) of $\mathcal{H}_3$ are written in Dirac notation as $|0\rangle$, $|1\rangle$, and $|2\rangle$, where $|0\rangle = (1,0,0)^T$, $|1\rangle = (0,1,0)^T$, and $|1\rangle = (0,0,1)^T$. An arbitrary vector $|\Psi\rangle$ in $\mathcal{H}_3$ can be expressed as a linear combination $|\Psi\rangle = a_0\,|0\rangle + a_1\,|1\rangle + a_2\,|2\rangle$, $a_0, a_1, a_2 \in \mathbf{C}$ and $|a_0|^2 + |a_1|^2 + |a_2|^2 = 1$. The real number $|a_i|^2$ is the probability that the state vector $|\Psi\rangle$ will be in i^{th} basis state upon measurement. Note that the basis vectors in the computational basis are ordered by natural numbers. The state space of ternary quantum system of $n > 1$ qutrits is a composite complex vector space formed from the algebraic tensor product $\mathcal{H}_3^{\otimes n}$ of component state spaces $\mathcal{H}_3$. The computational basis for $\mathcal{H}_3^{\otimes n}$ consists of all possible tensor products of the computational basis vectors of the component spaces; each vector in this basis consists of column vectors with the entry 1 in the i-th row and zeros in all others, where i ranges from 1 to n. [17] An arbitrary vector $|\Psi\rangle$ in $\mathcal{H}_3^{\otimes n}$ can be expressed as linear combination of the basis vectors with scalars $a_i \in \mathbf{C}$ such that $\sum_{i=0}^{n-1} |a_i|^2 = 1$. The real number $|a_i|^2$ is the probability that the state vector $|\Psi\rangle$ will be in i^{th} basis state upon measurement.

The evolution of an n qutrit quantum system occurs via the action of a linear operator that changes the state vector via multiplication by a $3^n \times 3^n$ unitary *evolution* matrix. From a computational point of view, the evolution matrices are quantum logic gates transforming the state vectors in $\mathcal{H}_3^{\otimes n}$. From a quantum logic synthesis point of view, these gates need to be implemented by a universal set of quantum gates. It is a well-known established fact that sets of one- and two- qutrit (and in general, *qudit*) gates are universal [10,1]. Hence, logic synthesis requires that $3^n \times 3^n$ evolution matrices be efficiently decomposed to the level of one and two qutrit gates. There are several methods from matrix theory, such as QR factorization, that have been utilized for this purpose in binary quantum logic synthesis. Another method is the Cosine-Sine Decomposition (CSD) of an arbitrary unitary matrix described in section 2. This method has been recently used by Mottonen [8] et. al, and Shende [7] et. al for binary logic synthesis. Recently, Bullock et.al have given a synthesis method for multi-valued quantum logic gates using a variation of the QR matrix factorization [3]. This paper presents a CSD based method for ternary quantum logic synthesis.

F.S.K.: E-mail: faisal@pdx.edu

2. THE COSINE-SINE DECOMPOSITION (CSD)

The Cosine-Sine decomposition has been used recently [8, 7] in the synthesis of binary quantum gates, which are $2^n \times 2^n$ unitary matrices for n qubit gates. When used in conjunction with local optimization techniques, the CSD provides a recursive synthesis method with a lower number of elementary gates compared to other methods [7].

Cosine-Sine Decomposition: [9, 7] Let the unitary matrix $W \in \mathbf{C}^{m\times m}$ be partitioned in 2×2 block form as

$$W = \begin{array}{c c} & \begin{array}{cc} r & m-r \end{array} \\ \begin{array}{c} r \\ m-r \end{array} & \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \end{array} \tag{1}$$

with $2r \leq m$. Then there exist $r \times r$ unitary matrices U_1, V_1, $r \times r$ real diagonal matrices C and S, and $(m-r) \times (m-r)$ unitary matrices U_2, V_2 such that

$$W = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} \begin{pmatrix} C & -S & 0 \\ S & C & 0 \\ 0 & 0 & I_{m-2r} \end{pmatrix} \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix} \tag{2}$$

The matrices C and S are the so-called cosine-sine matrices and are of the form C=diag($\cos\theta_1$, $\cos\theta_2$,$\ldots$, $\cos\theta_r$), S=diag($\sin\theta_1$, $\sin\theta_2, \ldots, \sin\theta_r$), such that $\sin^2\theta_i + \cos^2\theta_i = 1$ for $1 \leq i \leq r$.

CSD for Binary Quantum Logic Synthesis: In case of binary quantum logic, all matrices are even dimensional as powers of two. Hence, a given $m \times m$ unitary W can be always partitioned into $m/2 \times m/2$ square blocks, giving $m/2 \times m/2$ square matrices U_1, U_2, V_1, V_2, C, S upon application of the CSD. The decomposition for this case is given in equation (3),

$$W = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} \begin{pmatrix} C & -S \\ S & C \end{pmatrix} \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix} \tag{3}$$

The CSD can be applied to the 2×2 block diagonal factors that occur at each iteration until one reaches the qubit level which involves only 2×2 matrices [8]. At each iteration level the block diagonal matrices in the decomposition are realized as *quantum multiplexers* [7]. A quantum multiplexer is a gate acting on k+1 qubits of which one is designated as the control qubit. If the control qubit is the highest order qubit, the multiplexer matrix is block diagonal. Depending on whether the control qubit carries $|0\rangle$ or $|1\rangle$, the gate then performs either the top left block or the bottom right block of the $(k+1) \times (k+1)$ block diagonal matrix on the remaining k bits.

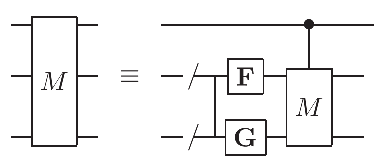

Figure 1. A 3-qubit quantum multiplexer. The / represents two wires, one for each lower qubit. Depending on the value of the controlling qubit, **F** or **G** is applied to the lower two qubits.

For instance, a quantum multiplexer matrix for 3 qubits will be a 2×2 block diagonal matrix with each block matrix of size 4×4 given in equation (4). The value of the first qubit is $|0\rangle$ in the location of the block matrix

F and $|1\rangle$ in the location of the block matrix **G**. Therefore, depending on whether the control bit carries $|0\rangle$ or $|1\rangle$, the gate then performs either **F** or **G** on the remaining 2 qubits respectively.

$$\begin{pmatrix} \mathbf{F} & 0 \\ 0 & \mathbf{G} \end{pmatrix} \tag{4}$$

The cosine-sine matrices in the CSD are realized as *uniformly k-controlled* R_y rotations [7, 8]. Such gates operate on $k+1$ qubits, of which the lower k are controls and the top one is the target. A different R_y is applied to the target for each control bit-string. The circuit for a uniformly 2-controlled R_y rotation is given in figure 2.

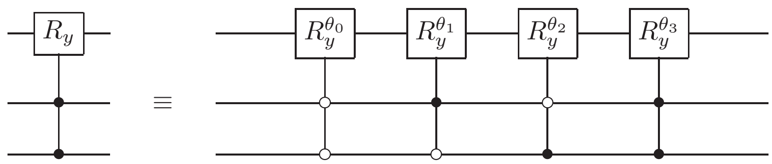

Figure 2. A uniformly 2-controlled R_y rotation: the lower two bits are the control bits, and the top bit is the target bit. In general, it requires 2^k one qubit controlled gates to implement a uniformly k-controlled rotation. The open circles represent the value $|0\rangle$ and the closed circles the value $|1\rangle$.

For three qubits $|a\rangle, |b\rangle$, and $|c\rangle$, the matrix formulation of a uniformly 2-controlled R_y rotation gate is given in equation (5).

$$\begin{pmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \otimes \begin{pmatrix} b_0 \\ b_1 \end{pmatrix} \otimes \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} \tag{5}$$

As $|b\rangle$ and $|c\rangle$ in equation (5) take on the values $|0\rangle$ and $|1\rangle$ in four possible combinations, θ_i takes on the values from the set $\{\theta_0, \theta_1, \theta_2, \theta_3\}$, resulting in different R_y gates being applied to the top most qubit. Each θ_i is an arbitrary angle.

3. CSD FOR TERNARY QUANTUM LOGIC SYNTHESIS

For ternary quantum logic synthesis, an n-qutrit gate will be a unitary matrix W of size $3^n \times 3^n$. Partition W as in (1) with $m = 3^n$ and $r = 3^{n-1}$, so that $m - r = 3^n - 3^{n-1} = 3^{n-1}(3-1) = 3^{n-1} \cdot 2$. After the application of the CSD, W will take the form in equation (2). The matrix blocks U_2, V_2 will be of size $3^{n-1} \cdot 2 \times 3^{n-1} \cdot 2$; hence an application of the CSD only on these two blocks will decompose each block into the form in equation (3). After these two application of the CSD and some matrix factoring, W will take the form

$$W = \Sigma \begin{pmatrix} C & -S & 0 \\ S & C & 0 \\ 0 & 0 & I \end{pmatrix} \Gamma \tag{6}$$

with

$$\Sigma = \begin{pmatrix} X_1 & 0 & 0 \\ 0 & X_2 & 0 \\ 0 & 0 & X_3 \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & C_1 & -S_1 \\ 0 & S_1 & C_1 \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & Z_1 & 0 \\ 0 & 0 & Z_2 \end{pmatrix} \tag{7}$$

and

$$\Gamma = \begin{pmatrix} Y_1 & 0 & 0 \\ 0 & Y_2 & 0 \\ 0 & 0 & Y_3 \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & C_2 & -S_2 \\ 0 & S_2 & C_2 \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & W_1 & 0 \\ 0 & 0 & W_2 \end{pmatrix} \tag{8}$$

Each block matrix in the decomposition given in equations (6) - (8) above is of size $3^{n-1} \times 3^{n-1}$. We realize each block diagonal matrix as a ternary quantum multiplexer acting on n qutrits of which the highest order qutrit is designated as the control qutrit. Depending on which of the values $|0\rangle$, $|1\rangle$, or $|2\rangle$ the control qutrit carries, the gate then performs either the top left block, the middle block, or the bottom right block respectively on the remaining $n-1$ qutrits. The cosine-sine matrices with identity in top-left/bottom-right block corner are realized as uniformly $(n-1)$-controlled R_x/R_z rotations. These matrices can be realized as R_x or R_z rotation [6] matrices in $\mathbf{R}^3$ applied to the top most qutrit, controlled by the lower qutrits as they range over $\{|0\rangle, |1\rangle, |2\rangle\}$. Each configuration of the lower qutrits leads to a different R_x or R_z gate.

EXAMPLE Consider two qutrits being acted upon by an arbitrary gate Q. The CSD synthesis of Q is given in figure 3.

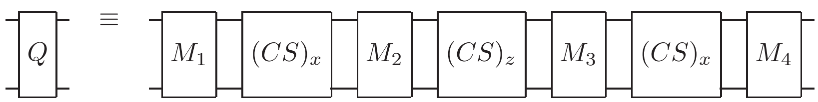

Figure 3. The decomposition of an arbitrary 2-qutrit gate Q using the CSD. Each M_i is a ternary quantum multiplexer. The gates $(CS)_x$ and $(CS)_z$ are uniformly 1-controlled rotations

For $1 \leq i \leq 4$, each M_i gate in figure 3 is a quantum multiplexer controlled by the top qutrit and can be decomposed to the level of elementary gates as shown in figure 4.

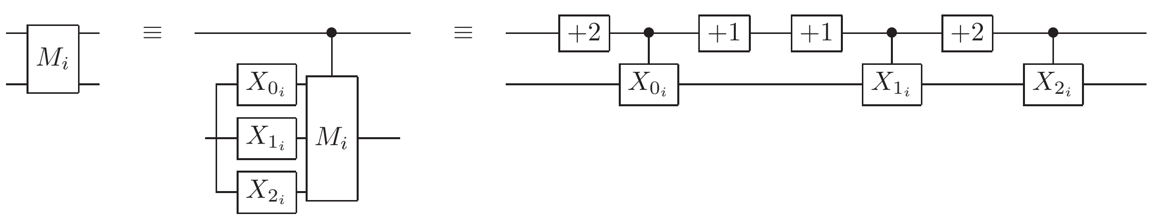

Figure 4. Quantum Ternary Multiplexer for second qutrit and its realization in terms of Muthukrishan-Stroud gates. The gates labled $+1$ and $+2$ are bit shifts increasing the value of the bit by 1 and 2 mod 3 respectively. Depending on the value of the top control qutrit a, one of X_{a_i} is applied to the second qutrit, for $a \in \{0, 1, 2\}$.

For two qutrits, the matrix for a ternary quantum multiplexer will be a 3×3 block diagonal matrix given in equation (9). The value of the first qutrit is $|0\rangle$ in the location of the block matrix $\mathbf{F}$, $|1\rangle$ in the location of the block matrix $\mathbf{G}$, and $|2\rangle$ in the location of the block matrix $\mathbf{H}$. Therefore, depending on whether the control bit carries $|0\rangle$, $|1\rangle$, or $|2\rangle$, the gate then performs either $\mathbf{F}$, $\mathbf{G}$, or $\mathbf{H}$ on the remaining qutrit respectively. All blocks in the matrix in equation (9)are of size 3×3.

$$\begin{pmatrix} \mathbf{F} & 0 & 0 \\ 0 & \mathbf{G} & 0 \\ 0 & 0 & \mathbf{H} \end{pmatrix} \tag{9}$$

The gates $(CS)_x$ and $(CS)_z$ in figure 3 are uniformly 1-controlled R_x and R_z rotations respectively. In either case, the top qutrit is controlled by the lower one, as shown in figure 5. The gate $(CS)_z$ corresponds to the middle matrix in equation (6). The matrix formulation of this gate as a uniformly 1-controlled R_z rotation is given in equation (10).

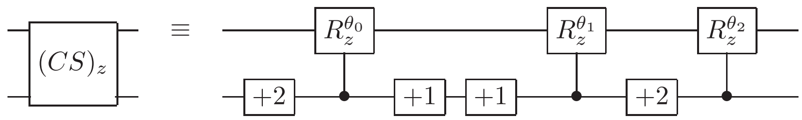

Figure 5. Uniformly 1-controlled rotations for 2-qutrits, realized as multiplexers via Muthukrishan-Stroud gates. The gates labeled +1 and +2 are bit shifts modulo 3.

$$\begin{pmatrix} \cos\theta_i & -\sin\theta_i & 0 \\ \sin\theta_i & \cos\theta_i & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} \otimes \begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix} \tag{10}$$

Depending on the three possible binary configurations of $|b\rangle$, θ_i takes on the values from the set $\{\theta_0, \theta_1, \theta_2\}$, resulting in different R_z gates being applied to the top qutrit. If the lower qutrit is $|0\rangle$, equation (10) reduces to

$$\begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{11}$$

If the lower qutrit is $|1\rangle$ or $|2\rangle$, equation (10) reduces to equations (12) and (13) respectively.

$$\begin{pmatrix} \cos\theta_2 & -\sin\theta_2 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{12}$$

$$\begin{pmatrix} \cos\theta_3 & -\sin\theta_3 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{13}$$

Hence, a 2-qutrit quantum gate can be synthesized via four 1 qutrit quantum multiplexers and three 1-qutrit uniformly controlled rotations on the first qutrit. In general, an n-qutrit quantum gate can be synthesized via four $n-1$ qutrit quantum multiplexers and three uniformly $n-1$ controlled rotations on the top qutrit.

4. CONCLUSIONS AND FUTURE WORK

We give a recursive procedure for ternary quantum logic synthesis by realizing n qutrit logic gates as $3^n \times 3^n$ unitary matrices and applying the Cosine-Sine Decomposition. We conclude that this method can synthesize a n qutrit gate with four multiplexers acting on $n-1$ qutrits and three uniformly $n-1$-controlled rotations. A two qutrit example is given. It is our future goal to do a gate count by investigating local optimizations at each level of recursion. We also intend to write a CAD tool for this decomposition and get a gate count for a higher number of qutrits, and extend the decomposition to odd radix multi-valued quantum logic synthesis.

5. ACKNOWLEDGMENTS

F. S. Khan is grateful to Jacob Biamonte for discussions, advice, and help in the layout of this paper. The Quantum Circuit diagrams were all drawn in LaTeX using Q-circuit available at http://info.phys.unm.edu/Qcircuit/.

REFERENCES

1. A. Muthukrishnan, C. R. Stroud Jr, *Multi-Valued Logic Gates for Quantum Computation*, Revised version to appear in Physical Review A. `quant-ph/0002033 v2`
2. M. H. A. Khan, M. A. Perkowski *Evolutionary Algorithm Based Synthesis of Multi-Output Ternary Functions Using Quantum Cascade of Generalized Ternary Gates*
3. S. Bullock, Dianne P. OLeary, Gavin K. Brennen *Asymptotically Optimal Quantum Circuits for d-level Systems*, `quant-ph/0410116 v2`
4. H. B. Pasquinucci, A. Peres, *Quantum Cryptography with 3-state Systems* `quant-ph/0001083 v1`
5. D. L. Zhou, B. Zeng, Z. Xu, C. P. Sun, *Quantum Computation Based on d-Level Cluster State* `quant-ph/0304054 v2`
6. K. Fuji, *Quantum Optical Construction of Generalized Pauli and Walsh-Hadamard Matrices in Three Level Systems* `quant-ph/0309132 v1`
7. V. Shende, S. Bullock, I. Markov, *Synthesis of Quantum Logic Circuits*, To appear,IEEE Transactions on Computer Aided Design. `quant-ph/0406176`
8. M. Mottonen, J. J. Vartiainen, V. Bergholm, M. M. Salomaa, *Quantum circuits for general multiqubit gates*, Phys. Rev. Lett. 93, 130502 (2004). `quant-ph/0404089`
9. G. W. Stewart, *On the Perturbation of Pseudo-Inverses, Projections and Linear Least Square Problems*,SIAM Review, Vol. 19, N0 4 (Oct. 1977), 634-662, Appendix.
10. R. K. Brylinski, G. Chen, *Mathematics of Quantum Computation* Chapman Hall/CRC, 2002, ISBN: 1584882824.
11. G. H. Golub, Charles F. Van Loan, *Matrix Computations* John Hopkins University Press, 1989, ISBN: 0-8018-3772-3.
12. S. D. Bartlett, H. D. Guise, B. Sanders, *Quantum encodings in spin systems and harmonic oscillators*, Physical Review A, Vol. 65, 052316.
13. M. Miller, D. Maslov, G. Dueck, *Synthesis of Quantum Multiple-Valued Circuits*, Journal of Multiple-Valued Logic and Soft Computing, special issue on Nano MVL Structures (24 journal pages, accepted).
14. M H. A. Khan, M. A. Perkowski, M. Khan, P. Kerntopf, *Ternary GFSOP Minimization using Kronecker Decision Diagrams and Their Synthesis with Quantum Cascades*, submitted to special issue of International Journal on Multiple-Valued Logic and Soft Computing, T. Hanyu, editor, 2004.
15. W. N. N. Hung, X. Song, G. Yang, J. Yang, M. A. Perkowski, *Quantum logic synthesis by symbolic reachability analysis*, In Proceedings of the 41st Design Automation Conference, San Diego, CA, June 2004.
16. A. Barenco, C. Bennet, R. Cleve, D. P. DiVincenzo, N. Margolus, P. Shor, T. Sleator, J. Smolin, H. Weinfurter, *Elementary gates for quantum computation*, Physical Review A, 52:3457, 1995. `quant-ph/9503016 v1`
17. A. N. Al-Rabadi, *Reversible Logic Synthesis: From Fundamental to Quantum Computing*, Springer-Verlag, Berlin, Heidelberg 2004, ISBN: 3-540-00935-3
18. D. C. Marinescu,G. M. Marinescu, *Approaching Quantum Computing*, Pearson Education, Inc. 2005, New Jersey, ISBN: 0-13-145224-X