Bi-Decomposition of Multi-Valued Relations

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June 13, 2001

Motivation and Conclusion

- Multi-Valued relations are used in
  - machine learning
  - data mining
  - MV circuits
- Bi-Decomposition of Multi-Valued relations using MAX- and MIN-Gates generates *compact* and *well-balanced* circuits
- using BEMDDs, the algorithm is very *fast*

BEMDDs

- Two ways to encode ten values using binary variables
- Truth table for relation R(A,B)

	00	01	11	10
00	0	4	8	8
01	1	5	9	9
11	3	7	10	10
10	2	6	10	10

	00	01	11	10
00	0	2	7	4
01	0	2	8	4
11	1	3	10	6
10	1	3	9	5

B\A	0	1
0	0,1,2	2
1	1	0,1,2
2	1,2	0,1

BEMDDs

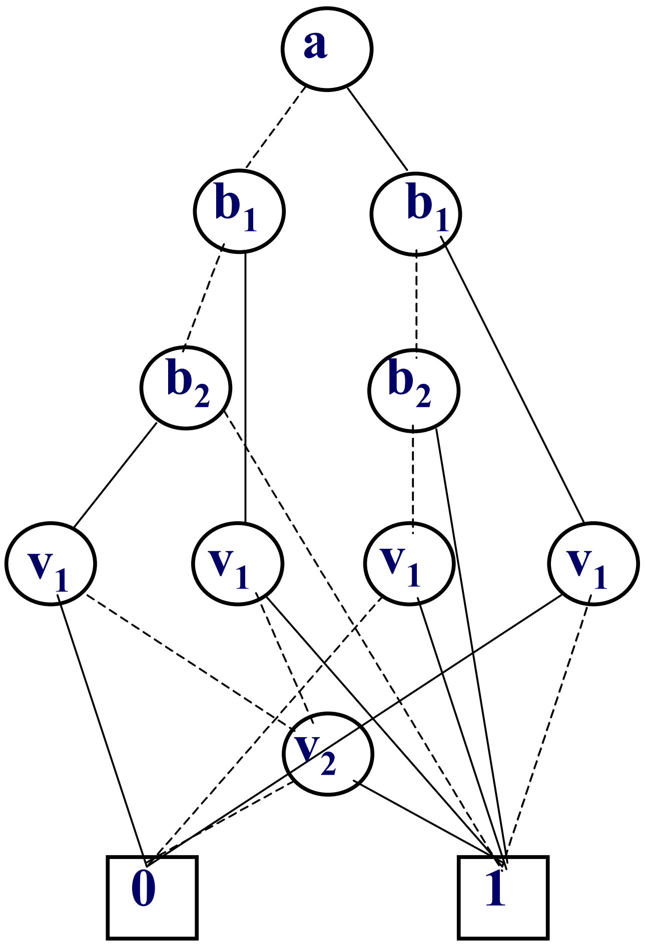
- The Karnaugh map for the binary relation encoding multi-valued relation R(A,B)

$v_1v_2 \backslash ab_1b_2$	000	001	011	010	110	111	101	100
00	1	0	0	0	1	1	1	0
01	1	1	1	1	1	1	1	0
11	1	0	1	1	0	0	1	1
10	1	0	1	1	0	0	1	1

$$R(a,b_1,b_2,v_1,v_2) = \bar{a}\bar{b}_1\bar{b}_2 + \bar{a}\bar{b}_1 b_2 \bar{v}_1 v_2 + \bar{a} b_1 (v_1 + v_2) + a\bar{b}_1\bar{b}_2 v_1 + a\bar{b}_1 b_2 + a b_1 \bar{v}_1$$

BEMDDs

Binary
Encoded
Multi-Valued
Decision
Diagrams



Operations on MV Relations

- Interval Relation
- A multi-valued relation is an *interval relation* if in each vertex (minterm) of the domain, the output values form a contiguous range.

B\A	0	1
0	0,1,2	2
1	1	0,1,2
2	1,2	0,1

Operations on MV Relations

- "Less than" - Relation

$$R_{v<w}(v,w)$$

$v \backslash w$	0	1	2	3
0	0	1	1	1
1	0	0	1	1
2	0	0	0	1
3	0	0	0	0

- "More than" - Relation

$$R_{v>w}(v,w)$$

$v \backslash w$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	1	1	0	0
3	1	1	1	0

Operations on MV Relations

- Lower and Upper Bound Intervals

R

B\A	0	1
0	0,1	2
1	1	0,1,2
2	1,2	0,1

R_{LBI}

B\A	0	1
0	0	0,1,2
1	0,1	0
2	0,1	0

R_{UBI}

B\A	0	1
0	1,2	2
1	1,2	2
2	2	1,2

$$R_{LBI}(x, v) = \overline{\exists_v [R(x, v) \,\&\, R_{v<w}(v, w)]_{w \to v}},$$

$$R_{UBI}(x, v) = \overline{\exists_v [R(x, v) \,\&\, R_{v>w}(v, w)]_{w \to v}},$$

Operations on MV Relations

- Interval Increments and Decrements for Lower Bound Relations

R_{LBI}

B\A	0	1
0	0	0,1,2
1	0,1	0
2	0,1	0

R_{LBI}^{U+}

B\A	0	1
0	0,1	0,1,2
1	0,1,2	0,1
2	0,1,2	0,1

R_{LBI}^{U-}

B\A	0	1
0	-	0,1
1	0	-
2	0	-

$$R_{LBI}^{U-}(x, v) = \exists v \left[R_{LBI}(x, v) \,\&\, R_{v>w}(v, w) \right]_{w \to v} ,$$

$$R_{LBI}^{U+}(x, v) = \overline{\exists v \left[\overline{R_{LBI}(x, v)} \,\&\, R_{v<w}(v, w) \right]_{w \to v}} =$$

$$= \forall v \left[R_{LBI}(x, v) + R_{v \geq w}(v, w) \right]_{w \to v} .$$

Operations on MV Relations

- Interval Increments and Decrements for Upper Bound Relations

R_{UBI}

B\A	0	1
0	1,2	2
1	1,2	2
2	2	1,2

R_{UBI}^{L+}

B\A	0	1
0	2	2
1	2	2
2	2	2

R_{UBI}^{L-}

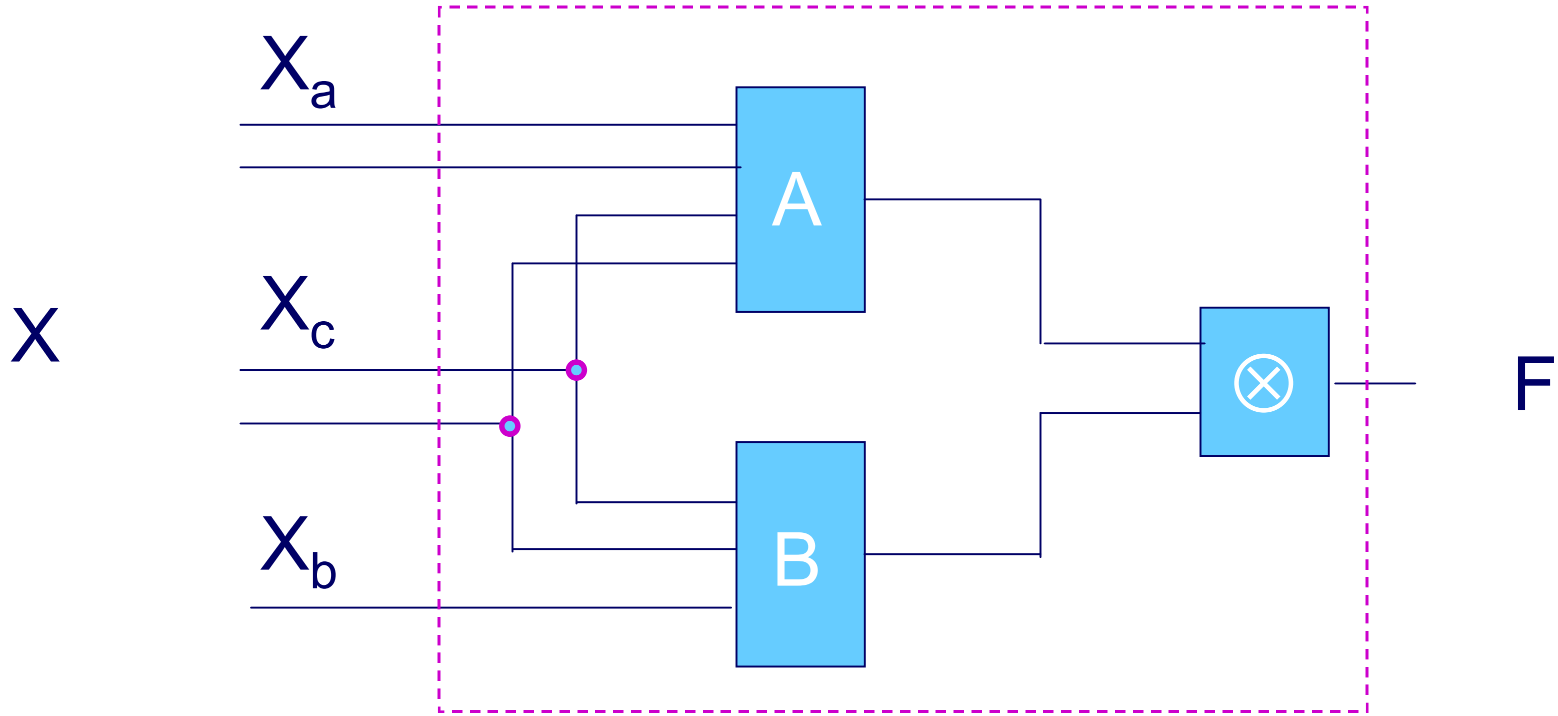
B\A	0	1
0	0,1,2	1,2
1	0,1,2	1,2
2	1,2	0,1,2

$$R_{UBI}^{L-}(x, v) = \exists v \left[R(x, v)_{UBI} \ \& \ R_{v<w}(v, w) \right]_{w \to v} ,$$

$$R_{UBI}^{L+}(x, v) = \overline{\exists v \left[\overline{R_{UBI}(x, v)} \ \& \ R_{v>w}(v, w) \right]_{w \to v}} =$$

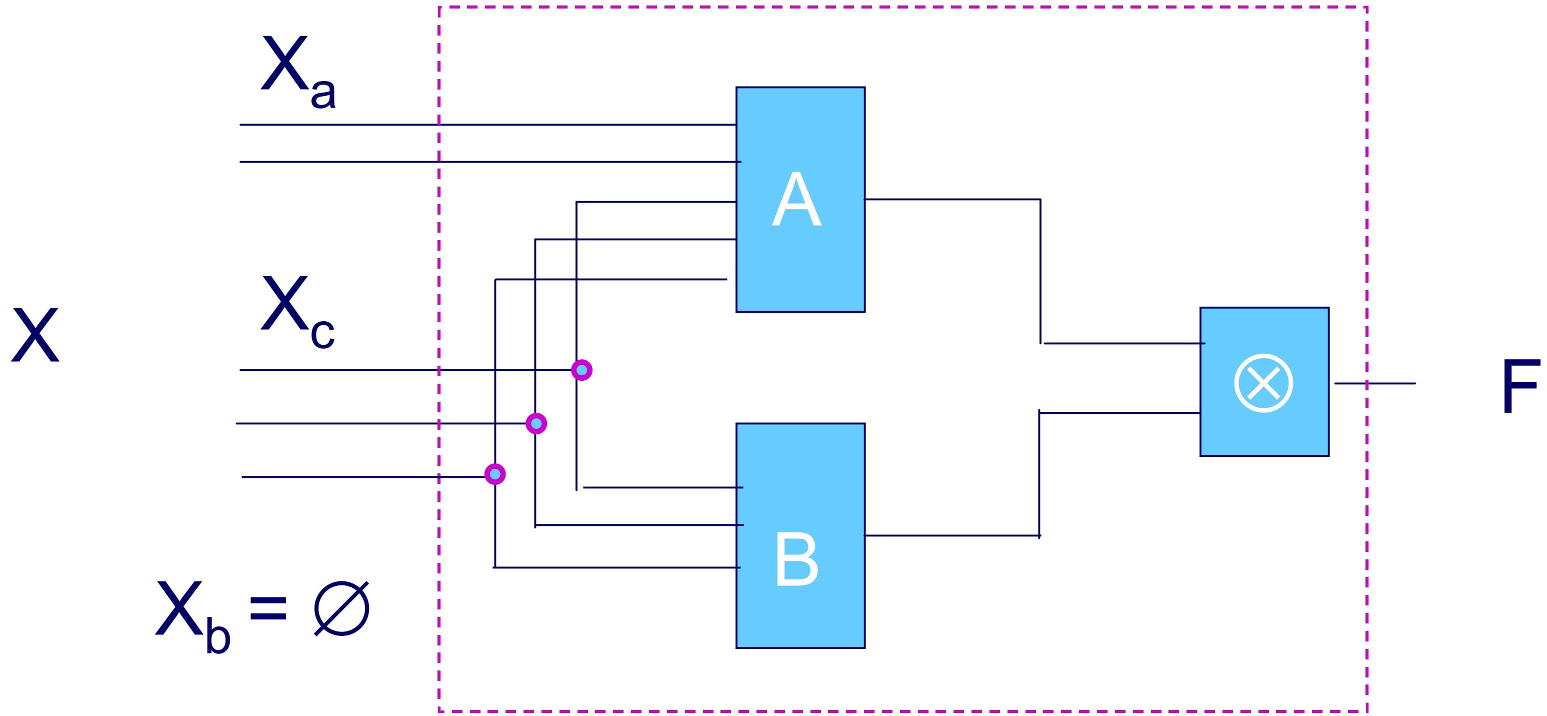
$$= \forall v \left[R_{UBI}(x, v) + R_{v \le w}(v, w) \right]_{w \to v} .$$

Strong Multi-Valued Bi-Decomposition



$$F(X) = GATE(\ A(X_a, X_c),\ B(X_b, X_c)\ )$$

Weak Multi-Valued Bi-Decomposition



$$F(X) = GATE(\ A(X_a, X_c),\ B(X_c)\ \ )$$

Checking for MV MAX-Bi-decomposition

- Critical Relation: $R_{\text{crit}}(x,v) = [R_{\text{LBI}}(x,v)]^{\text{U-}}$

- **Theorem.** Multi-valued interval relation $R(x,v)$ specified by UBI $R_{\text{UBI}}(x,v)$ and LBI $R_{\text{LBI}}(x,v)$ has strong MAX-bi-decomposition with variable sets (x_a, x_b) iff

$$R_{\text{crit}}(x,v) \;\&\; \exists x_a\, R_{\text{UBI}}(x,v) \;\&\; \exists x_b\, R_{\text{UBI}}(x,v) = 0$$

Checking for MV MAX-Bi-decomposition

R

B\A	0	1
0	1,2	0,1
1	0,1,2	0
2	2	1,2

R_{LBI}

B\A	0	1
0	0,1	0
1	0	0
2	0,1,2	0,1

R_{UBI}

B\A	0	1
0	2	1,2
1	2	0,1,2
2	2	2

R_{crit} &
$\exists x_a\ R_{UBI}$ &
$\exists x_b\ R_{UBI} = 0$

$R_{crit} = [R_{LBI}]^{U-}$

B\A	0	1
0	0	-
1	-	-
2	0,1	0

$\exists x_a\ R_{UBI}$

B\A	0	1
0	1,2	1,2
1	0,1,2	0,1,2
2	2	2

$\exists x_b\ R_{UBI}$

B\A	0	1
0	2	0,1,2
1	2	0,1,2
2	2	0,1,2

B\A	0	1
0	-	-
1	-	-
2	-	-

R^A

B\A	0	1
0	2	0
1	2	0
2	2	0

R^B

B\A	0	1
0	1	1
1	0	0
2	2	2

$MAX(R^A, R^B)$

B\A	0	1
0	2	1
1	2	0
2	2	2

Checking for MV MAX-Bi-decomposition

- Weak MAX-Bi-decomposition
- **Theorem.** Multi-valued relation R(x,v) specified by its UBI $R_{UBI}(x,v)$ and LBI $R_{LBI}(x,v)$ is MAX-bi-decomposable in the weak sense with the variable sets (x_a, $x_b=\varnothing$) iff

$$\psi(x) = \exists v R_{crit}(x,v) - \exists v[R_{crit}(x,v) \;\&\; \exists x_a \; R_{UBI}(x,v)] \neq 0$$

Deriving decomposed MV Relations

Theorem. If the multi-valued relation R(x,v) specified by UBI $R_{UBI}(x,v)$ and LBI $R_{LBI}(x,v)$ is MAX-Bi-decomposable with the variable sets (x_a, x_b), the relations of blocks A and B are:

$$R^A_{LBI} = \exists x_b[(v = 0) + R_{LBI} \,\&\, \exists v\,(R_{crit} \,\&\, \exists x_a R_{UBI})\,]$$

$$R^A_{UBI} = \exists x_b\, R_{UBI}$$

$$R^B_{LBI} = \exists x_a[(v = 0) + R_{LBI} \,\&\, \exists v\,(R_{crit} \,\&\, R^A)\,]$$

$$R^B_{UBI} = \exists x_a\, R_{UBI}$$

Deriving decomposed MV Relations

$R_{crit} \& \exists x_a R_{UBI}$

B\A	0	1
0	-	-
1	-	-
2	-	-

R^{A}_{LBI}

B\A	0	1
0	0	0
1	0	0
2	0	0

R^{A}_{UBI}

B\A	0	1
0	2	0,1,2
1	2	0,1,2
2	2	0,1,2

R^{A}

B\A	0	1
0	2	0
1	2	0
2	2	0

$R_{crit} \& R^{A}$

B\A	0	1
0	-	-
1	-	-
2	-	0

R^{B}_{LBI}

B\A	0	1
0	0	0
1	0	0
2	0,1	0,1

R^{B}_{UBI}

B\A	0	1
0	1,2	1,2
1	0,1,2	0,1,2
2	2	2

R^{B}

B\A	0	1
0	0,1	0,1
1	0	0
2	1,2	1,2

$MAX(R^{A}, R^{B})$

B\A	0	1
0	2	0,1
1	2	0
2	2	1,2

MV Bi-decomposition algorithm

procedure **BiDecompose**(bdd LBI, bdd UBI)
{ bdd LBI_A, UBI_A, LBI_B, UBI_B, S, F_A, F_B, F;
(LBI, UBI) = **RemoveInessentialVariables**(LBI, UBI);
if (|S| < 2) { (F, gate) = **CreateConstantBlockMAXLiteral** (LBI, UBI);
AddBlockToDecompositionTree(F, gate);
return F; }
bdd X_A^{MAX}, X_B^{MAX}, X_A^{MIN}, X_B^{MIN}, X_A^{BEST}, X_B^{BEST};
(X_A^{MAX}, X_B^{MAX}) = **GroupVariablesMAX**(LBI, UBI);
(X_A^{MIN}, X_B^{MIN}) = **GroupVariablesMIN**(LBI, UBI);
(X_A^{BEST}, X_B^{BEST}, gate) = **FindBestVariableGrouping**((X_A^{MAX}, X_B^{MAX}), (X_A^{MIN}, X_B^{MIN}));
if ((X_A^{BEST}, X_B^{BEST}) == (∅, ∅)) (X_A^{BEST}, X_B^{BEST}, gate) = **GroupVariablesWeak**(LBI, UBI);
if ((X_A^{BEST}, X_B^{BEST}) == (∅, ∅)) return **CompletelySpecifiedFunction**(LBI, UBI);
(LBI_A, UBI_A) = **DeriveBlockA**(LBI, UBI, X_A^{BEST}, X_B^{BEST} , gate);
F_A = **BiDecompose**(LBI_A, UBI_A);
(LBI_B, UBI_B)=**DeriveBlockB**(LBI, UBI,F_A,X_A^{BEST},X_B^{BEST}, gate);
F_B = **BiDecompose**(LBI_B, UBI_B);
F = **Gate**(F_A, F_B);
AddBlockToDecompositionTree(F, gate);
return F;
}

Experimental Results

Bmark	In/Out	Val	Cubes	Bdd nodes	Reading time,c	Logic levels	DFC	Gates				BiDec time,c
								MM	Lits	NonDec	Total	
audiology	69/1	154	200	12677	0.27	15	54648	94	74	14	182	0.91
balance	4/1	20	625	179	0.02	20	3600	121	115	0	236	0.07
baloon1	4/1	8	16	6	0.01	2	8	1	2	0	0	0.01
breastc	9/1	90	699	4027	0.07	10	1284	51	38	7	96	0.17
bridges1	9/1	29	108	616	0.01	11	2483	47	38	5	90	0.06
bridges2	10/1	32	108	815	0.02	13	3070	58	46	7	111	0.10
car	6/1	21	1728	246	0.07	10	603	31	30	0	61	0.02
chess1	6/1	40	28056	15074	1.37	64	$1.7\cdot10^6$	5081	3493	716	9290	29.67
chess2	36/1	73	3196	9448	0.84	10	415	68	62	4	134	1.61
cloud	6/1	48	108	621	0.01	9	599	24	16	5	45	0.04
employ1	9/1	27	9600	154	0.51	12	505	26	23	0	49	0.01
employ2	7/1	29	18000	132	0.77	8	639	32	31	0	63	0.01
flag	28/1	133	194	10504	0.13	12	4313	96	75	13	184	0.51
flare1	10/3	33	969	305	0.06	8	1656	56	45	8	109	0.08
flare2	10/3	33	3198	342	0.17	12	9787	113	85	17	215	0.15
hayes	4/1	18	132	122	0.01	5	117	8	9	0	17	0.01
lung-c	56/1	224	32	4171	0.09	8	232	16	14	2	32	0.09
mushroom	22/1	117	8124	1230	1.18	6	64	5	6	0	11	0.03
programm	12/1	42	20000	47581	2.27	49	$3.6\cdot10^5$	11160	6621	1502	19283	64.46
sensory	11/1	36	576	3506	0.06	27	$7.9\cdot10^4$	633	442	109	1184	0.71
ships	4/1	16	34	105	0.01	9	320	15	10	2	27	0.01
sleep	9/1	83	62	1274	0.02	8	582	17	17	0	34	0.04
sponge	44/1	165	76	3146	0.09	5	80	5	6	0	11	0.06
tic-tac-toe	9/1	27	958	697	0.07	17	2134	274	208	42	524	0.24
train	32/1	105	10	336	0.01	2	8	1	2	0	3	0.01
zoo	16/1	39	101	448	0.02	6	431	8	9	0	17	0.01

Experimental Results

- Comparison of decomposition results with YADE

Bmark	YADE [5]			BI-DECOMP-MV		
	Gates	DFC	Time, c	Gates	DFC	Time, c
balance	268	2012	18	236	3600	0.09
breastc	95	634	24	96	1284	0.24
flare1	154	932	11	109	1656	0.14
flare2	300	8049	37	215	9787	0.32
hayes	22	128	1	17	117	0.02