Quiz 2

Solutions

Please write all answers on these pages.

1. Consider the support vector machine \( h \) defined by the following set of support vectors and \( \alpha \)'s:

<table>
<thead>
<tr>
<th>Example</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( \alpha )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_1 )</td>
<td>0</td>
<td>1</td>
<td>.75</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>4</td>
<td>3</td>
<td>.25</td>
</tr>
<tr>
<td>( x_3 )</td>
<td>0</td>
<td>5</td>
<td>-.2</td>
</tr>
<tr>
<td>( x_4 )</td>
<td>2</td>
<td>6</td>
<td>-.25</td>
</tr>
</tbody>
</table>

and bias = 3.

(a) Give the weight vector of the separating line defined by these values.

\[
\mathbf{w} = .75(0,1) + .25(4,3) - .2(0,5) - .25(2,6) = (0,.75) + (1,.75) - (0,1) - (.5,1.5) = (.5,-1)
\]

(b) Give the equation of the line of the separating hyperplane defined by these values.

\[
.5x_1 - x_2 + 3 = 0
\]

or

\[
x_2 = 5x_1 + 3
\]

(c) Using dot products with support vectors, along with the bias, show how the new example \( x_5 = (1, 4) \) would be classified by the support vector machine \( h \).

\[
h(1,4) = sgn[.75(0,1) \cdot (1,4) + .25(4,3) \cdot (1,4) - .2(0,5) \cdot (1,4) - .25(2,6) \cdot (1,4) + 3]
\]

\[
= sgn[.75(4) + .25(16) - .2(20) - .25(26) + 3]
\]

\[
= sgn[1 + 4 - 4 - 6.5 + 3] = -1
\]
(d) Using the graph below, plot the support vectors and the separating hyperplane. Also draw a line to show the margin. Finally, plot the new point (1, 4) to confirm that it is in the class you found in part (c).

2. In class, the SVM optimization problem was stated as follows:

Given training examples $x_k$ each with target $t_k$, find $w$ and $b$ by doing the following minimization:

$$\min_{w,b} \frac{1}{2} \|w\|^2, \text{ subject to } \begin{cases} w \cdot x_k + b \geq 1 \text{ if } t_k = 1 \\ w \cdot x_k + b < -1 \text{ if } t_k = -1 \end{cases}$$

(a) Explain in a few sentences: why do we want to minimize $\frac{1}{2} \|w\|^2$ ?

We showed in class that the margin is equal to $\frac{1}{\|w\|}$. Thus to maximize the margin, we want to minimize $\|w\|$. To minimize, we need to take the derivative, so to make the math cleaner we minimize $\frac{1}{2} \|w\|^2$, which yields the same result as minimizing $\|w\|$. 
(b) Explain in one or two sentences: what is the reason for the constraint in the minimization problem:

subject to \[ \begin{align*}
\mathbf{w} \cdot \mathbf{x}_k + b &\geq 1 \text{ if } t_k = 1 \\
\mathbf{w} \cdot \mathbf{x}_k + b &< -1 \text{ if } t_k = -1
\end{align*} \]

This constraint forces the training examples to be classified correctly and to be outside of the margin. This constraint is relaxed for soft-margin SVMs.

3. The plot below demonstrates that the XOR problem is not linearly separable in two dimensions.

Will the mapping \( \Phi(x_1, x_2) = (x_1, x_2, (x_1 - x_2)^2) \) make these data points linearly separable in three dimensions? Explain or use a picture to show why or why not.

Here is the result of applying the mapping to the four data points:

(0, 1)\( \rightarrow \) (0, 1, 1)
(1, 0)\( \rightarrow \) (1, 0, 1)
(0, 0)\( \rightarrow \) (0, 0, 0)
(1, 1)\( \rightarrow \) (1, 1, 0)
The mapping allows the points to be separated by a plane perpendicular to the \( x_3 \)-axis as shown below:

[Image of a 3D coordinate system with points labeled and a plane shown]

4. Given the polynomial kernel \( k(\mathbf{x}, \mathbf{y}) = [(\mathbf{x} \cdot \mathbf{y}) + 1]^3 \) and the following training set, give the kernel matrix \( \mathbf{K} \) defined by this kernel and this training set.

Training set:
\( \mathbf{x}_1 = (0, 0) \)
\( \mathbf{x}_2 = (1, 2) \)
\( \mathbf{x}_3 = (2, 3) \)

\[
\mathbf{K} = \begin{pmatrix}
  k(\mathbf{x}_1, \mathbf{x}_1) & k(\mathbf{x}_1, \mathbf{x}_2) & k(\mathbf{x}_1, \mathbf{x}_3) \\
  k(\mathbf{x}_2, \mathbf{x}_1) & k(\mathbf{x}_2, \mathbf{x}_2) & k(\mathbf{x}_2, \mathbf{x}_3) \\
  k(\mathbf{x}_3, \mathbf{x}_1) & k(\mathbf{x}_3, \mathbf{x}_2) & k(\mathbf{x}_3, \mathbf{x}_3)
\end{pmatrix}
\]

\[
= \begin{pmatrix}
  ((0,0) \cdot (0,0) + 1)^3 & ((0,0) \cdot (1,2) + 1)^3 & ((0,0) \cdot (2,3) + 1)^3 \\
  ((1,2) \cdot (0,0) + 1)^3 & ((1,2) \cdot (1,2) + 1)^3 & ((1,2) \cdot (2,3) + 1)^3 \\
  ((2,3) \cdot (0,0) + 1)^3 & ((2,3) \cdot (1,2) + 1)^3 & ((2,3) \cdot (2,3) + 1)^3
\end{pmatrix}
\]

\[
= \begin{pmatrix}
  1 & 1 & 1 \\
  1 & 216 & 729 \\
  1 & 729 & 2744
\end{pmatrix}
\]