CS 410/584, Algorithm Design & Analysis, Lecture Notes 8

Computational Geometry

Algorithms for manipulation of geometric objects

We will concentrate on 2-D geometry

Numerically robust – try to avoid division

- Round-off error
- Divide-by-0 checks

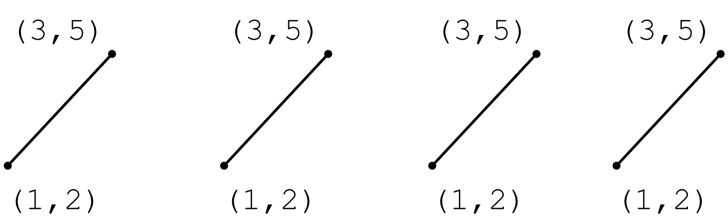
Techniques

Basics

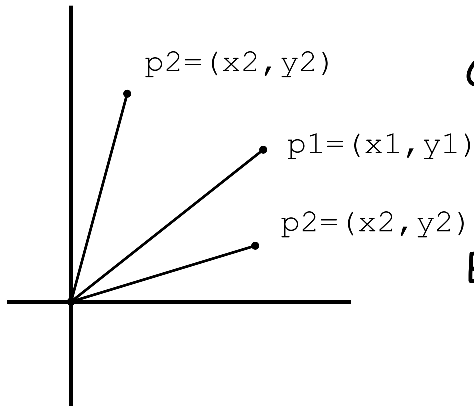
Point p =

Line segment p_1–p_2

Can represent different things



Relative Orientation of Segments



How do we know which case it is?

Could compare slopes

But that has division problems

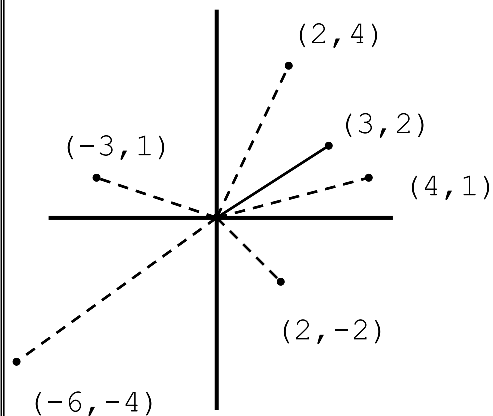
Cross Product

Can compare slopes without division

y1/x1 < y2/x2

x2·y1 < x1·y2

Examples



$(3,2)\times(2,4)=$

$(3,2)\times(4,1)=$

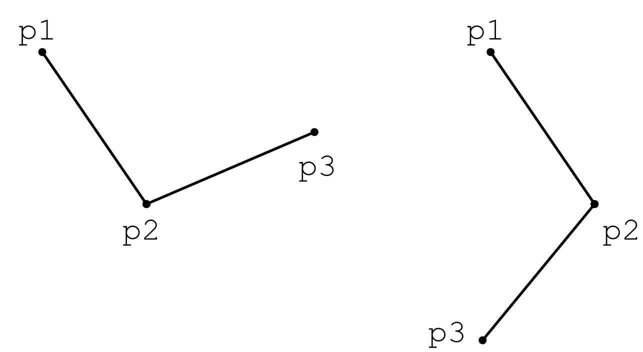
$(3,2)\times(2,-2)=$

$(3,2)\times(-3,1)=$

$(3,2)\times(-6,-4)=$

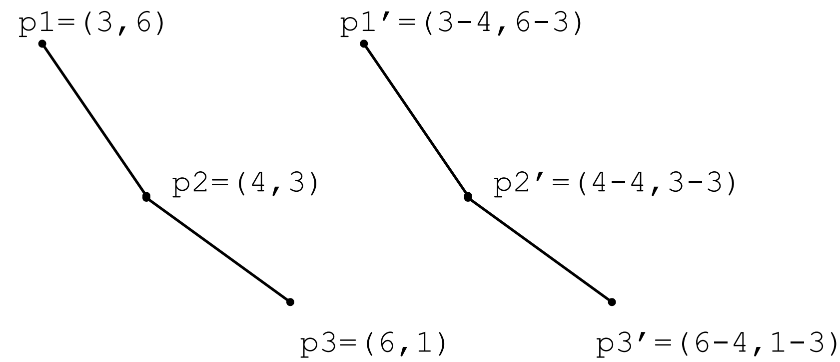
Left Turn, Right Turn

How do we know if a sequence of two line segments turns to left or right?

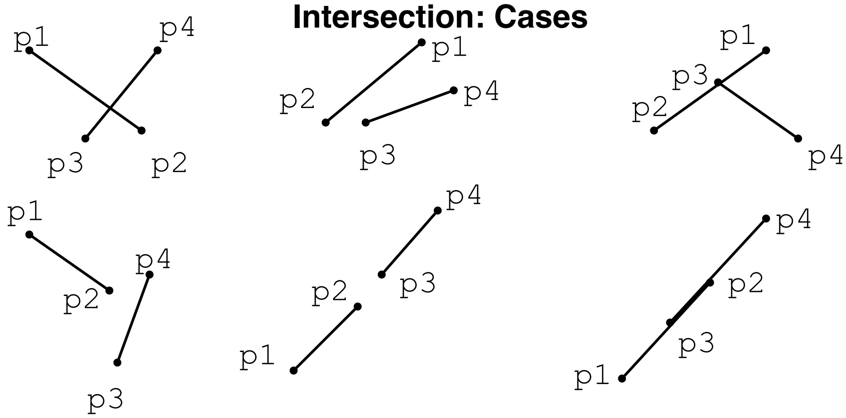


Use Cross Product

Translate p2 to origin, along with other points. Look at relative position of p1, p3



Intersection: Cases



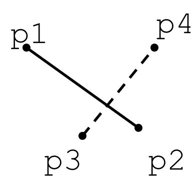
Main conditions

p1, p2 on different sides of

p3, p4 on different sides of

How Do You Tell Different Sides?

One is a left turn and one is a right turn



If 0 Change in Direction

Then the point is on the p1—p2 line somewhere

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OnSegment(p1,p2,p3)
1. min(x1,x2) ≤ x3 ≤ max(x1,x2)
2. min(y1,y2) ≤ y3 ≤ max(y1,y2)
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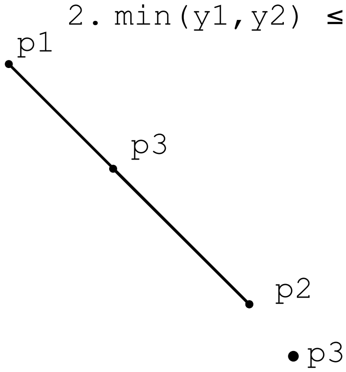


Beware of Boundary Conditions

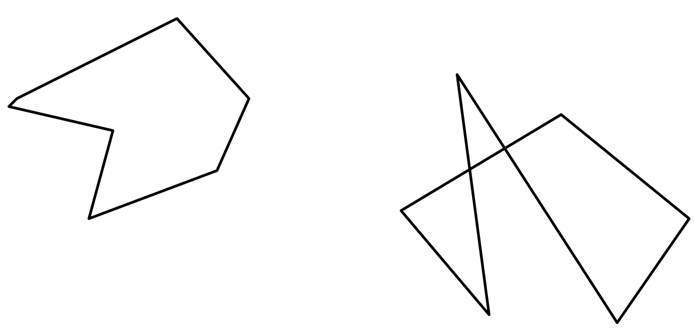
- Co-linear points, points on line
- Multiple intersections
- Vertical, horizontal lines
- p1—p2 versus p2—p1
- Degenerate objects

  What about intersection with the degenerate line segment p3—p3

Polygon

Polygon – closed sequence of sides

<u>Simple</u> polygon if edges don't cross



Exercise 33.1-7

Find out if point p is in simple polygon P

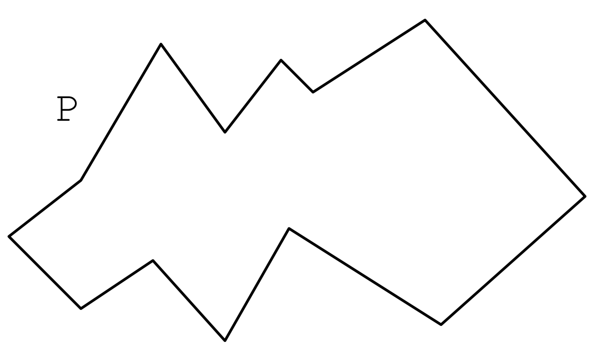
P has n points p1, p2, …, pn, want O(n)

1. Assume no 3 consecutive points on the same line
2. Make sure p doesn't lie on any edge of P
3. Construct a segment p—q where q is for sure
4. Count number of segments of P that intersect p—q

How to Find a Point Outside of P

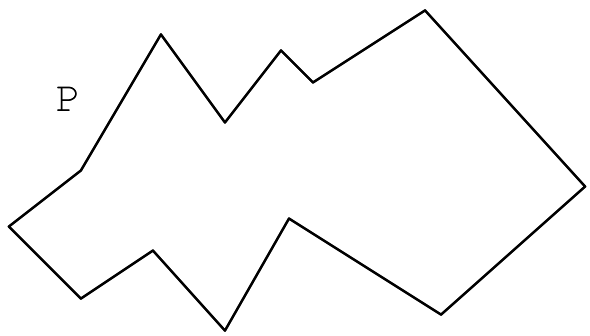
Let x′ = max xi

If p = (x, y), let q =



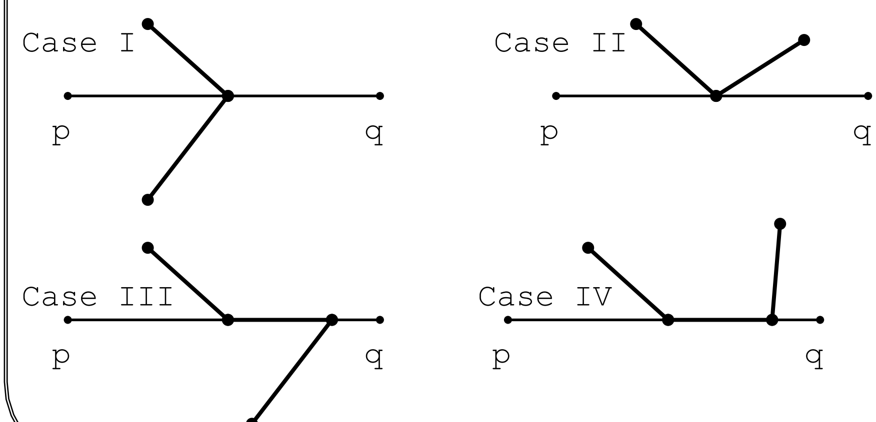
Cases to Watch out For

p—q passes through a vertex



What’s the Difference in These Cases

Do previous and next points lie on the same side of p—q or not?



In Cases III & IV, need to test if p_{i-1} and p_{i+2} are same side of p—q or not

In O(n) time, can head around polygon, counting number of intersections with p—q, taking into account

Pairwise Intersection

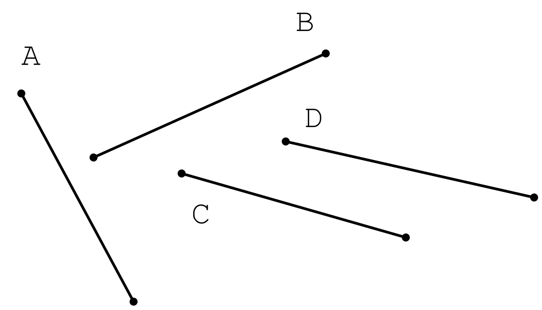
Have a collection of *n* line segments, want to know if there is a pair that intersect

Perhaps they represent traces on a circuit layer

Sweep = A vertical line moving left to right over the segments (conceptually)

Interested in Intersections with Sweep Line

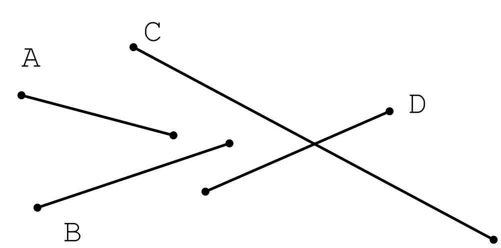
Want the intersection points in top-to-bottom order



Intersecting Segments

Intersecting segments adjacent in some sweep list

Assumptions:
- No 3 segments intersect
- No vertical segments

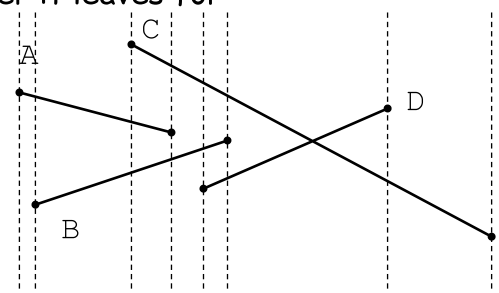


Where Does Sweep List Change?

Sweep list changes at segment endpoints.

Update sweep list "just after" encountering each endpoint

- after it enters for
- after it leaves for



Maintaining Sweep List

Keep sweep list as a structure that supports O(log n) operations

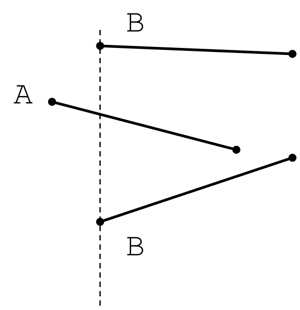
Operations:

- Insert segment
- Delete segment
- Above:
- Below:

Some kind of balanced tree, ordered on

How to Order New Segment on Sweep Line

Need to know where new segment B is relative to each segment A already in sweep line.



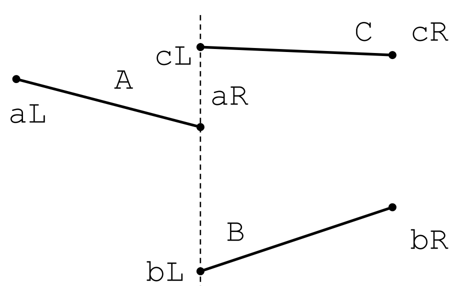
Order Endpoint List

Need to have the endpoints in the order that the sweep line encounters them

Sort on x-coordinate

What about ties?

- Left endpoints win over
- Lower y-coordinate wins over



Algorithm

EP (endpoints):

SL (sweep list)

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for each p in EP in order
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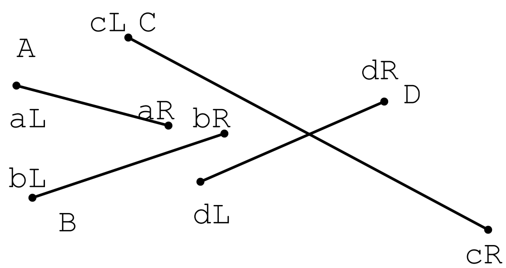
- insert s into SL
- check if s intersects lines above and below

- check if segments above and below s intersect each other
- delete s from SL

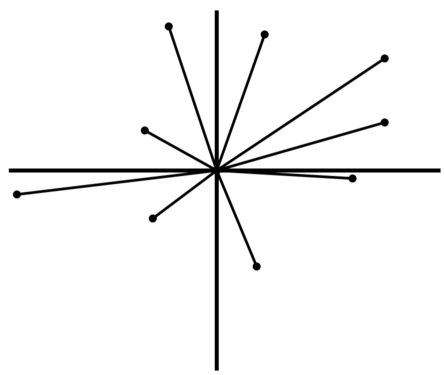
Example



EP	SL
aL	
bL	
cL	
aR	
dL	
bR	
dR	
aR	

Polar Angle

Next couple algorithms, need to order points by polar angle.



Discussion Problems

Have Q = {p1, p2, …, pn}

1. Given p, find point pi in Q with smallest positive polar angle from p.
2. Sort points in Q by increasing polar angle from x axis.

Try to solve these problems using only cross product.

Convex Hull

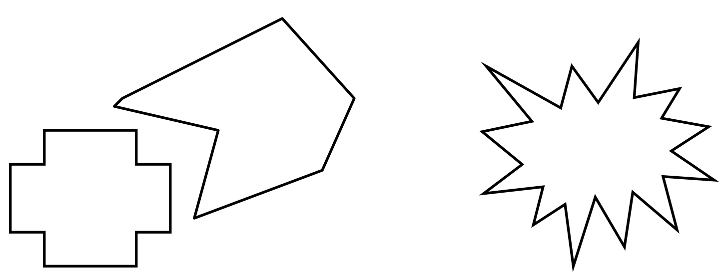
Given Q = {p1, p2, …, pn}, what is smallest convex polygon P such that

Vertices of P will be in Q

Consider Q with 100 points.
- What is the most that can be on convex hull?
- What is the fewest that can be on convex hull?

Uses of Convex Hull

Approximate more complicated shape by convex hull, for intersection "pre-check"

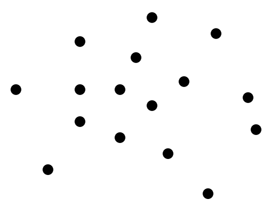


Multiple Convex Hull Algorithms

Some algorithms are good when the number of points on the hull is low. Others perform same on any number of points.

Can often use heuristics to speed up either.

Make any convex polygon R using points in Q.
Then points on interior of R

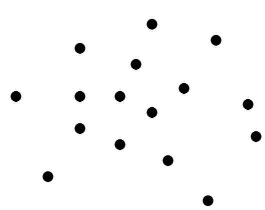


One Way to Choose R

Let pa, pb, pc, pd be the points with min x, max x, min y, max y.

Let R be polygon pa—pd—pb—pc—pa

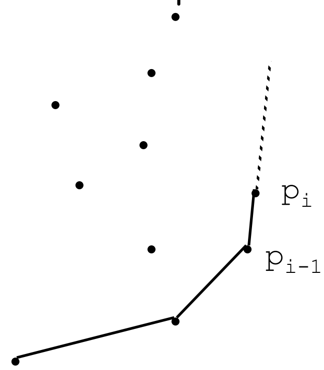
Will this always remove points?



Jarvis's March

Package wrapping:

Of points not on hull so far, which is at least angle with the last point on the hull



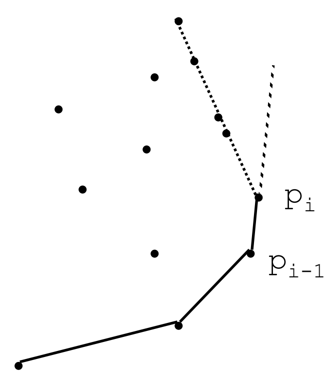
Issues

Co-linear points: What if several points on same edge of convex hull?

Want most points:

Want least points:

How to find distance?



Issues, Continued

Where to start:

What line to use through first point:

Complexity

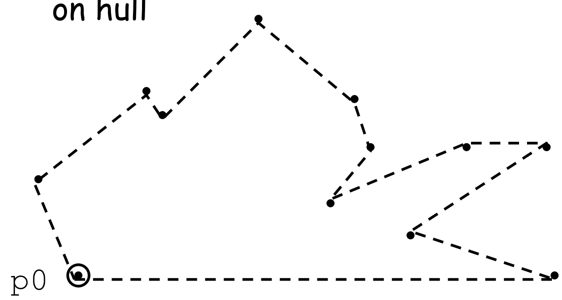
At each point, do an O(n) process to find point with minimum positive polar angle.

How many times do we need to do that search?

So, $O(n \cdot h)$ where h =

Graham Scan

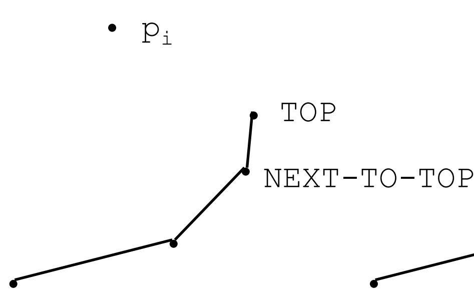
Start with “angle sweep” and remove points
- Find point p0 with minimum y-coordinate
- Sort points by polar angle from horizontal line through p
- Always O(n log n), no matter how many points on hull



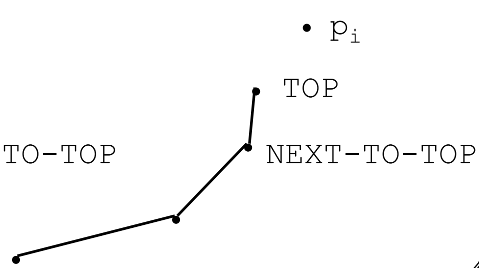
Candidate Hull on Stack

Push points on stack in order
- Before you push, check angle with top two points on stack

Left turn:

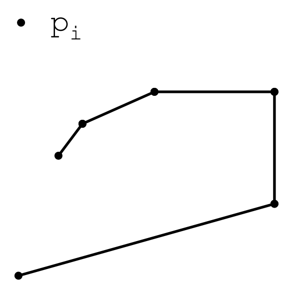


Right turn:



May be Several Pops in a Row

Can have several points popped because of pi



Graham Scan: Start-up

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Input:
1. Let p[0] have
2. sort p[1]...p[n] by angle
   with
3. Have stack S
4. Push(    )
   Push(    )
   Push(    )
```

Graham Scan: Body

```
for i = 3 to N
  while NEXT-TO-TOP—TOP—p[i] is

    POP
  PUSH(    )
S holds the convex hull when done
```


Complexity

O(n log n) to sort points

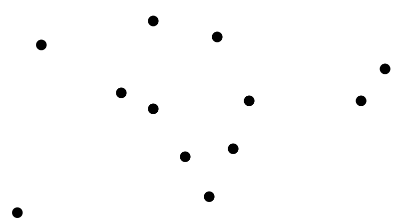
O(n) for rest

- Each point gets pushed once
- Can do multiple pops in while-loop, but each is of a previously pushed point

Planar Minimal Spanning Tree

Suppose I give you points in the plane to represent nodes in a graph

- All edges exist, weight is distance between end points.
- What is the first edge to add?

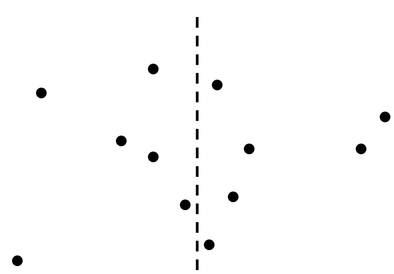


Closest Pair

Use divide and conquer

Split points in half by x-coordinate

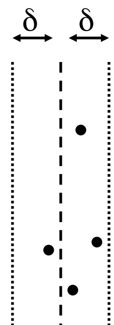
- Find closest pair on each side
- Let δ =



Closer Pair

Can there be a pair of points closer than δ?

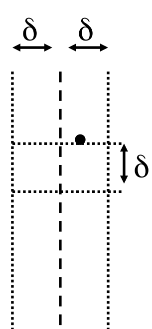
- Yes! Near the boundary
- So check



Avoid Too Many Comparisons

Don't want to compare all pairs of points near the boundary, otherwise get

- Only compare each point near boundary to next 7 in order of y-coordinate



Complexity

For n points, have two sub-problems of size n/2

How much other work outside of the recursive calls?

- Might seem like O(n log n) because of the sorting
- That would give complexity

However, make two lists of the points at the very beginning, one sorted on x and the other on y

- Can construct ordered lists for sub-problems in linear time.
- $T(n) = 2T(n/2) + O(n)$