CS 410/584, Algorithm Design & Analysis, Lecture Notes 6

Matrix Multiplication

For a divide-and-conquer approach, we want an n×n matrix to look like a matrix of smaller matrices

$$\begin{pmatrix} 1 & 3 & 21 & 9 \\ 4 & 12 & 36 & 35 \\ 42 & 6 & 10 & 53 \\ 2 & 32 & 12 & 60 \end{pmatrix} \begin{pmatrix} 41 & 2 & 6 & 14 \\ 3 & 11 & 7 & 18 \\ 40 & 13 & 16 & 29 \\ 7 & 19 & 63 & 72 \end{pmatrix}$$

How is a matrix of matrices like a matrix of integers?

Ring

(S, +, ·, 0, 1)

Axioms

1. +, · associate
2. + commutes
3. · distributes over +

4. 0 is identity for +
5. 1 is identity for ·
6. every a in S has an additive inverse

Example Ring

Integers modulo n, $\mathbf{Z}_n$

- $-[a]$ is
- 0 is
- 1 is

$\mathbf{Z}_4$

- $-[1]$ is
- $[1] + [3]$

Key Example

$$(M_n,\ +_n,\ \cdot_n,\ 0_n,\ 1_n)$$

is a ring.

Inverse of $A = (a_{ij})$ is

Get weird

- $R_{2,n/2}$ is

Going Between M_n and $R_{2,n/2}$

A, B, C in M_n where

A', B' C' in $R_{2,n/2}$

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} =$$

$$\begin{pmatrix} & \\ & \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

reduce n×n matrix mult to n/2×n/2 matrix ops

Complexity

Let M(n) be the number of scalar operations (   ) to multiply two n×n matrices over

Divide-and-conquer doesn't buy much

M(n) =

In general, if

M(n) =

then $M(n) \le kn^{\lg m}$ for some k,

provided m >

Complexity 2

If $m = 8$, as before

$M(n) = O(n^{\lg 8}) =$

If $m = 7$

$M(n) = O(n^{\lg 7})$

Need 2×2 multiplication with 7 scalar multiplications

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

e.g., $c_{22} =$

Strassen's Algorithm

$m_1 = (a_{12}-a_{22})(b_{21}+b_{22})$
$m_2 = (a_{11}+a_{22})(b_{11}+b_{22})$
$m_3 = (a_{11}-a_{21})(b_{11}+b_{12})$
$m_4 = (a_{11}+a_{12})b_{22}$
$m_5 = a_{11}(b_{12}-b_{22})$
$m_6 = a_{22}(b_{21}-b_{11})$
$m_7 = (a_{21}+a_{22})b_{11}$

Don't want to use commutativity of multiplication

More Algebra

$c_{11} = m_1 + m_2 - m_4 + m_6$

$c_{12} = m_4 + m_5$

$c_{21} = m_6 + m_7$

$c_{22} = m_2 - m_3 + m_5 - m_7$

$(a_{11}+a_{22})(b_{11}+b_{22})$

$- (a_{11}-a_{21})(b_{11}+b_{12})$

$+ a_{11}(b_{12}-b_{22})$

$- (a_{21}+a_{22})b_{11}$

Another Use of Matrix of Matrices

Connect matrix inversion with matrix multiplication

Inverse of matrix A, denoted A^{-1}, is the unique matrix such that $AA^{-1} = I$

- I is the identity matrix
- Inverse doesn't always exist

Example

$$\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} -3+4 & 2-2 \\ -6+6 & 4-3 \end{pmatrix}$$

Inversion vs. Multiplication

Assuming matrix inversion behaves reasonably, it has the same order as multiplication

Let I(n) =

1. I(n) in $\Omega(n^2)$
2. I(3n) in O(I(n))

Time to multiply two $n \times n$ matrices, M (n), is in O(I(n)).

Reduce Matrix Multiply to an Inversion Problem

Suppose we have A, B to multiply, where both are $n \times n$

Construct a $3n \times 3n$ matrix D

$$\begin{pmatrix} In & A & 0 \\ 0 & In & B \\ 0 & 0 & In \end{pmatrix}$$

Claim that D^{-1} is

$$\begin{pmatrix} In & -A & AB \\ 0 & In & -B \\ 0 & 0 & In \end{pmatrix}$$

Check

DD^{-1} =

$$\begin{pmatrix} I_n & A & 0 \\ 0 & I_n & B \\ 0 & 0 & I_n \end{pmatrix} \begin{pmatrix} I_n & -A & AB \\ 0 & I_n & -B \\ 0 & 0 & I_n \end{pmatrix}$$