Ford-Fulkerson Method

Flow maximization in a network (graph) with capacities

Basic idea:

- Find a path from source to target that still has flow capacity (*augmenting path*)
- Add the maximum flow allowed along this path
- Repeat until no augmenting path

Issues

1. How do we account for flow by cancellation
2. Does adding an augmenting path lead to a legal flow?
3. Will this process converge?
4. If so, will it lead to a maximum flow

Problem Formulation

Directed graph $G = (N, E)$

Two special nodes: s source t target

Assume for any node $v \in N$, there are paths

$s \rightsquigarrow v \rightsquigarrow t$

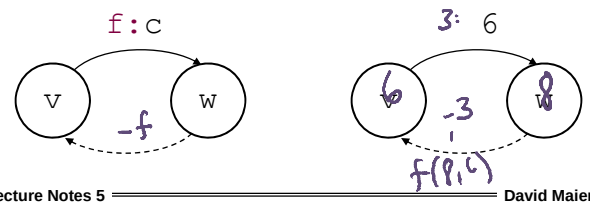
Capacity $c(v,w) \geq 0$

If (v, w) not an edge, then

$c(v,w) = 0$

Flow $f(v, w)$ can be

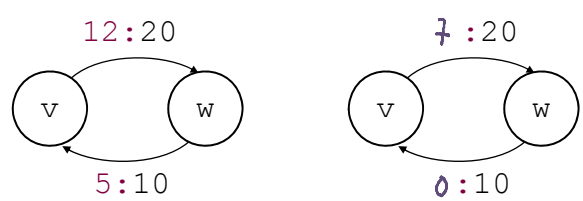
f(6,8) c(6,8)



Detail

Assume no "useless" flows between nodes

Positive flow in only one direction



Legal Flow f

1. $f(v, w) \leq c(v, w)$

2. $f(v, w) = -f(w, v)$

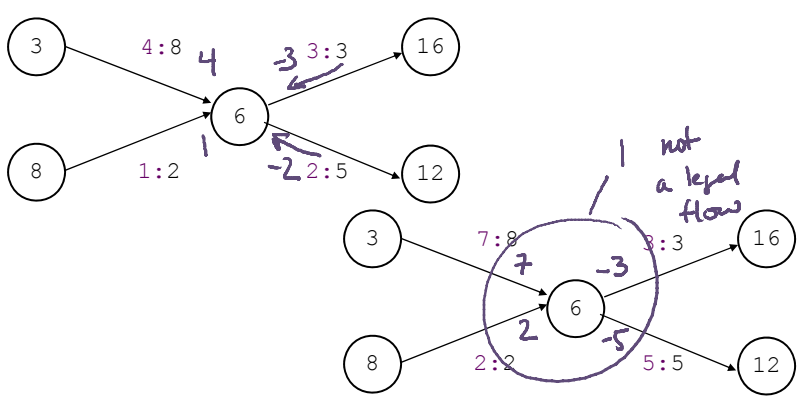
3. For any $v \neq s, t$

$$\sum_{w \neq v} f(v, w) = 0$$

Inputs = Outputs

Consider node 6

f(6,16) + f(6,12) + f(6,8) + f(6,3)

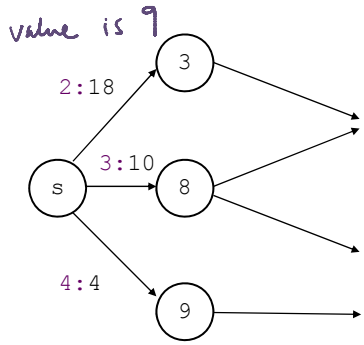


Value of Flow f

Total flow out of source

$$|f| = \sum f(s, w)$$

value is 9

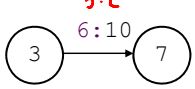


Residual Capacity for a Flow

Residual Capacity between nodes v and w:

$$r(u,v) = c(u,v) - f(u,v)$$

f:c



$$\underset{4}{r(3,7)} = \underset{10}{c(3,7)} - \underset{6}{f(3,7)}$$

$$\underset{6}{r(7,3)} = \underset{0}{c(7,3)} - \underset{-6}{f(7,3)}$$

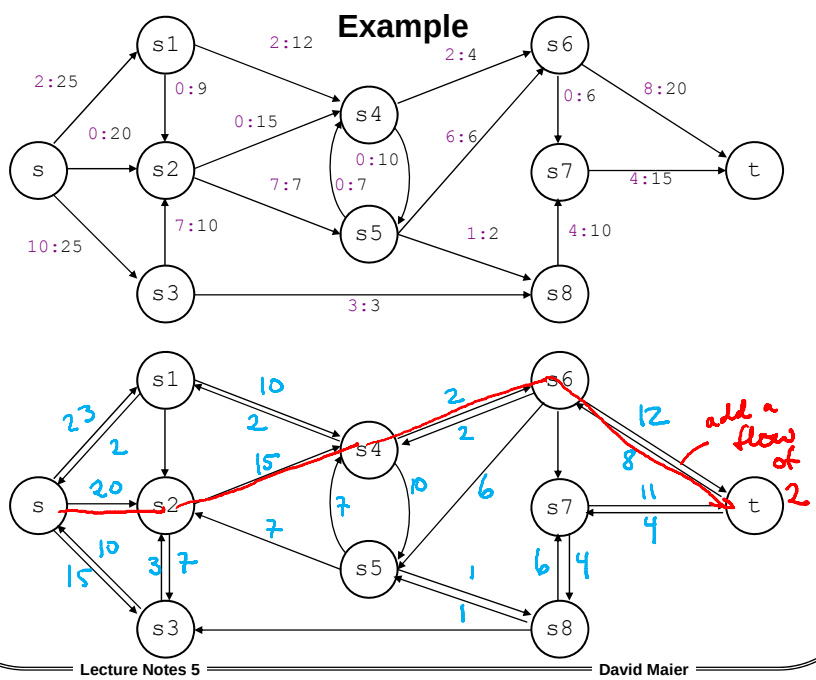
Residual Graph for Flow f

R = (N, E')

E' = {(v,w) | r(v,w) > 0 }

Capacities are residual capacities for f

Example

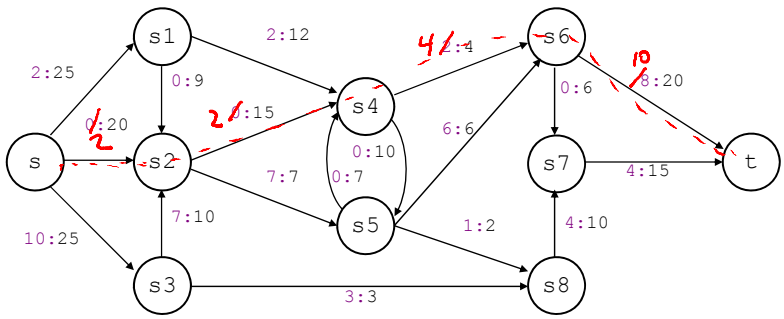


Capacity of a Path

Minimum capacity edge (v,w) r(v,w)

Add a flow of , along the path

r(v,w)



Is the Result a Legal Flow?

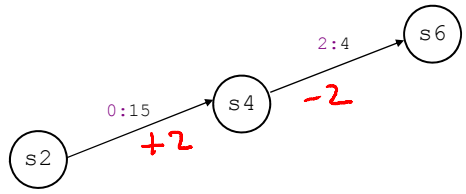
- Capacity?

residual graph shows available capacity

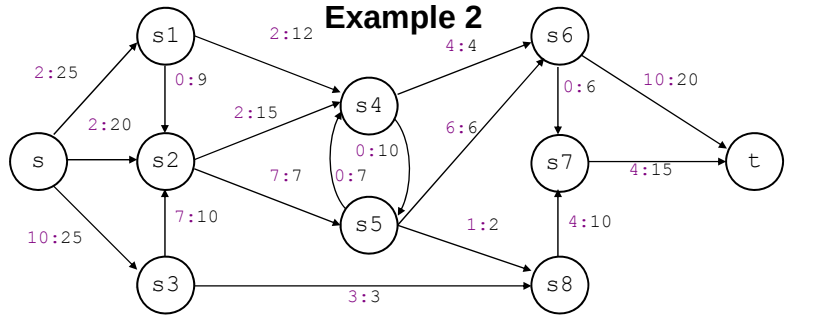
- Skew symmetry?

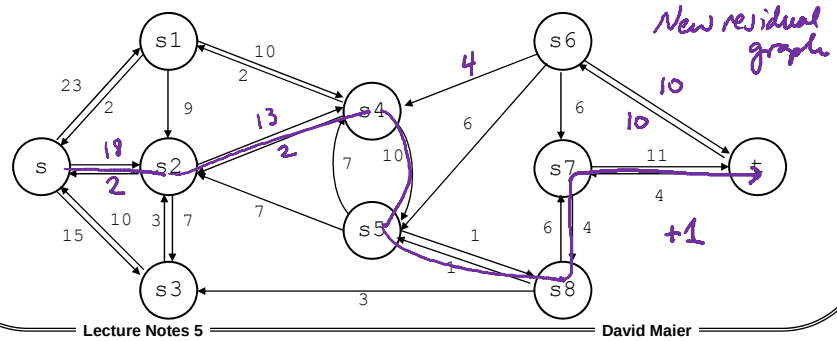
$f(v,w) = -f(w,v)$

- Conservation?

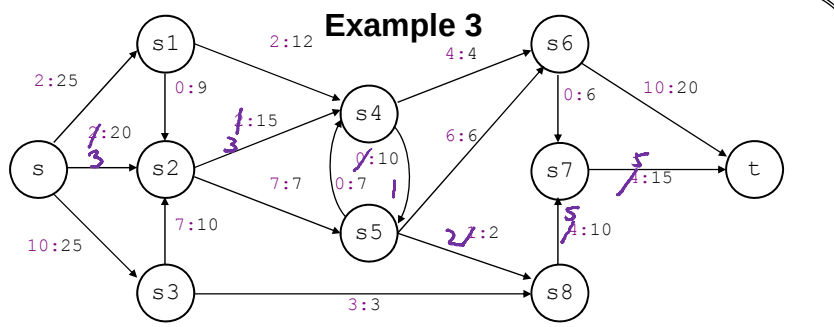


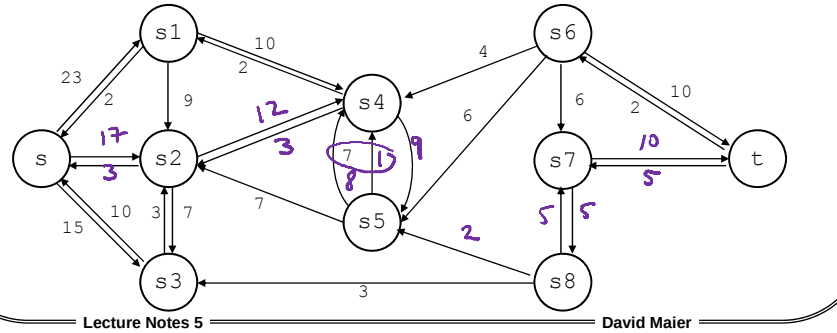
Example 2





Example 3

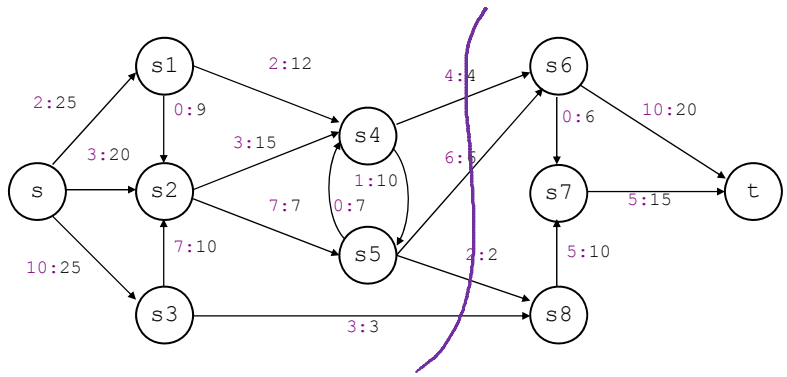




Is it a Maximum Flow?

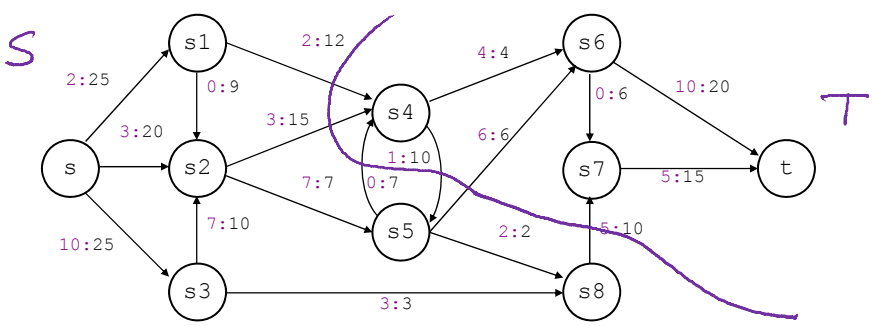
Would seem so:

Have a group of edges that divides s from t
and can't take more flow



Cut of a Graph

Divide nodes of N into two groups S, T (s → S, t → T)



Net flow across cut $\sum_{v \in S,\, w \in T} f(v, w)$ $= 2+3+0-1+6+5 = 15$

Capacity across cut $\sum_{v \in S,\, w \in T} c(v, w)$ $= 12+15+7+6+10 = 50$

Results

- Net flow across any cut $|f|$
- $|f|$ is bounded above by capacity of any cut
- Max-flow/min-cut theorem
  1. f is maximum flow
  2. residual graph has no augmenting paths
  3. $|f|$ is capacity of some $S|T$

2 ⇒ 3

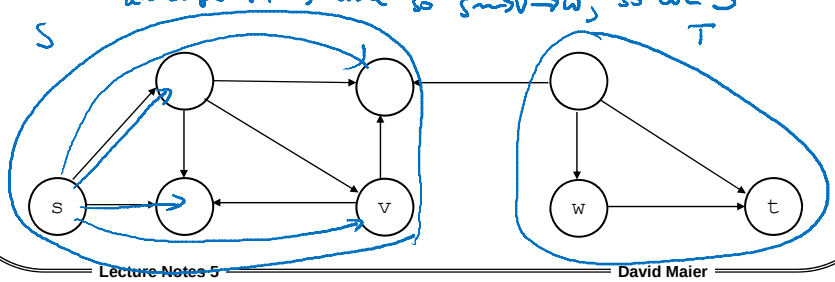
Consider residual graph R with no augmenting path

$S = \{v \mid s \rightsquigarrow v \text{ in } R\}$ $T = N - S$

Must have $t \in T$

Claim that for (v, w) with $v \in S$, $t \in T$, must have $r(v,w) = 0$ $(f(v,w) = c(v,w))$

Suppose not. Then $r(v, w) > 0$. Then R has an edge (v,w) and so $s \rightsquigarrow v \to w$, so $w \in S$



Basic Implementation

Start with 0 flow

Repeat

Add flow along an augmenting path

~~that~~ Until no such paths

Does it always converge?

Yes, if capacities are integers. Flow grows by at least 1 each time

How long does it take?

If you pick augmenting paths arbitrarily

$O(|E| \cdot |f^*|)$

f^* maximum flow in the graph

Edmonds-Karp Algorithm

Start with 0 flow

Repeat

Add flow along an augmenting path with shortest length

~~that~~ Until no paths

Time complexity no longer depends on value of maximum flow

$O(|N| \cdot |E|^2)$ time

Intuition: Length of shortest path to a node v in residual graph

- Each addition of flow increases distance (# of edges) to one node
- Distance to a node v can be increased at most |E| times, since |E| is length of longest possible path

Edmonds-Karp Example