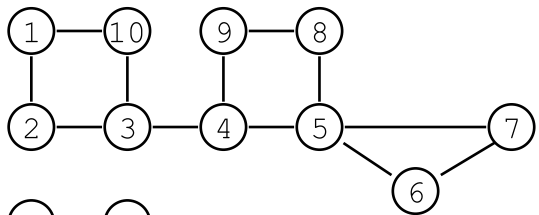
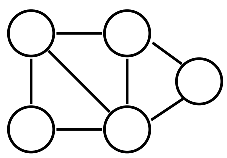
CS 410/584, Algorithm Design & Analysis, Lecture Notes 4

Biconnectivity

Let G = (N,E) be a connected

A node a ∈ N is an <u>articulation point</u> if there are v and w different from a such that every path from

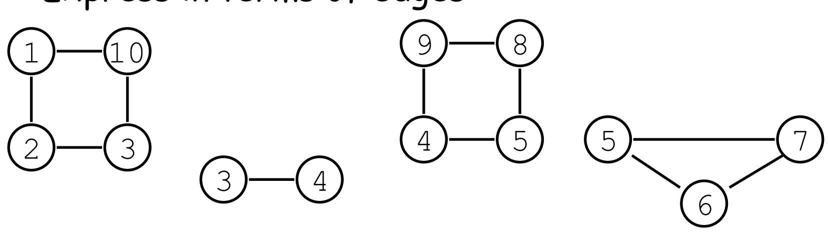




Biconnected Component

Maximal subgraph that is still connected after

Express in terms of edges



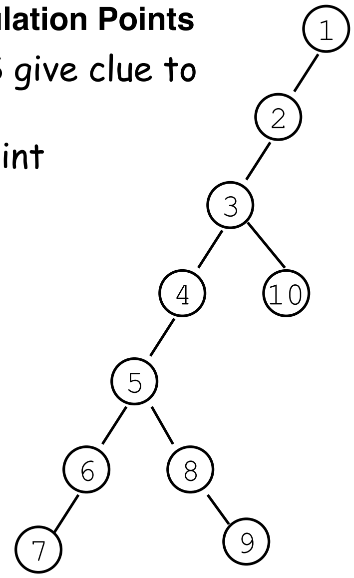
Articulation point must lie in two components

DFS and Articulation Points

Tree, back edges in DFS give clue to articulation points

If a is an articulation point

that splits v and w, then



DFS and Articulation Points 2

Lemma:

G = (N,E) is

S = (N,T) is

Node a is an articulation point if and only if

1. a is the root of S and

or

2. a is not the root and a has some child c in S where there is no

Low Numbers

For graph `G = (N,E)`, let

`T =`

`B =`

For simplicity, assume `DFNum[v] =`

How high (lowest `DFNum`) can we hop from a node on a back edge?

`LOW[v] =`

`min({v}` ∪

`{w|`there is `{x,w}` in `B` where

Calculating Low

If `{x,w}` in `B` and `w < v` then

Also, `v` is an articulation point if for some child `c` of `v`,

`LOW[c]`

Calculate `LOW` during `DFS`

`LOW[v]` is minimum of

1.

2.

3.

Biconnected Components Algorithm

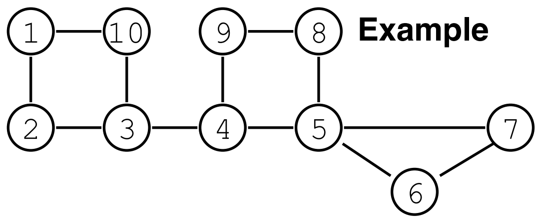
```
Count —
Parent[v] —
BI(v)
  mark v
  DFNum[v] ← Count, increment Count
  Low[v] ← DFNum[v]
```


Algorithm Cont.

```
for each edge {v,w} do
  if w not marked then



    if LOW[w] ≥ DFNum[v]
     and v not root then

    LOW[v] ←
  else if PARENT[v] ≠ w then
    LOW[v] ←
```

Need a check at end if root is an articulation point.

Example

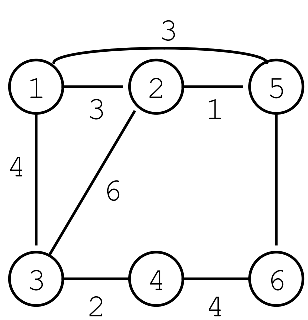


Minimum Cost Spanning Tree

G = (N, E)

<u>Cost function</u> c gives <u>edge weights</u>

A spanning tree is a graph S = ( ) where E E′ and S is a



Cost of a Spanning Tree

Cost of a spanning tree S = (N,E′) is

$\sum_{e \in E'}$

Want minimum cost

<u>Lemma</u>: Let S = (N,E′) be a spanning tree for G = (N,E). Then

a. for v_1, v_2 in N

b. adding an edge from E−E′ to S

Spanning Forest

A set of trees

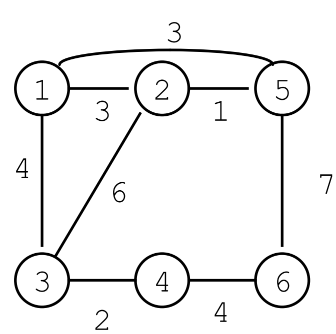
$\{(N_1,E_1),\ (N_2,E_2),\ \ldots,\ (N_k,E_k)\}$

Where

N =

$\subseteq$ E

E_i E_j



Expanding a Spanning Forest

Lemma: Suppose we have a spanning forest of $k > 1$ trees. Let $E' =$

Let $e = \{v,w\}$ be

such that $v \in N_1$, $w \notin N_1$.

Then there is a spanning tree for G that includes $E' \cup$ and has minimal cost among trees that include E'.

What kind of algorithm is this leading us towards?

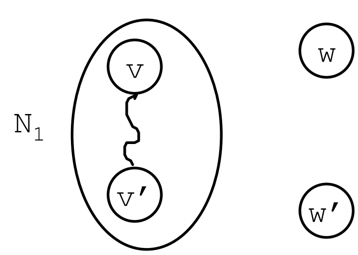
Proof of Lemma

Suppose not. Let

$S'' = (N,E'')$, $E' \subseteq E''$,

have lower cost than any spanning tree

Adding e to S'' causes



Proof of Lemma 2

S″ must have an edge e′ = {v′,w′} with

We know c(e) c(e′)

Let S = S″

S has no cycles

S is still a tree

Greedy Strategy

Take least cost edge e_1 in E. By last lemma,

[Consider an initial spanning forest

{ ({ },∅), ({ },∅), ..., ({ },∅) }]

Iterative Step

Let

$\{(N_1, E_1), (N_2, E_2), ..., (N_k, E_k)\}$

be a spanning forest that is

Add the lowest cost edge that

By the last lemma, the result is

Kruskal's Algorithm

```
KRUSKAL(G(N,E))
  S ⇐ ∅
  NS ⇐ {{v1},{v2},...,{vn}}
  Q ⇐
  while |NS| > 1 do
    {v,w}
    if Find(v) ≠ Find(w) then
      S ⇐ S
      Union(                )
```

Example

```
Q     c  NS
{2,5} 1  {1} {2} {3} {4} {5} {6}
{3,4} 2
{1,2} 3
{1,5} 3
{1,3} 4
{4,6} 4
{2,3} 6
{5,6} 7
```


Complexity

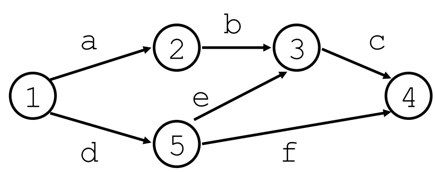
Priority Queue: sort, make a list

Union/Find
- # Union's =
- # Find's =

Can do a little better—
keep a heap of edges by cost

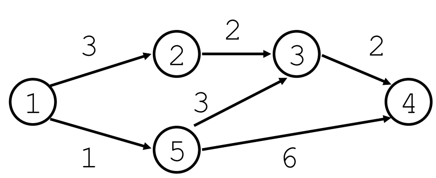
Path Problems

Directed graphs with labels on edges



Shortest Path

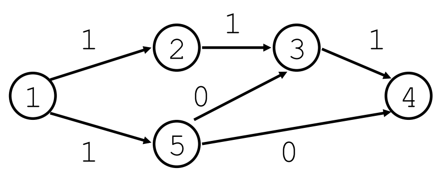
Labels could be non-negative integers
Use for



Connectivity

Labels could be Boolean values

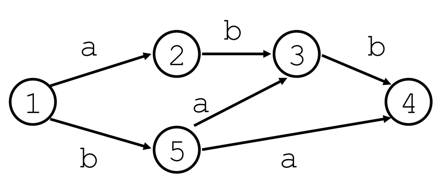
Use for



Finite Automata

Labels could be symbols from an alphabet

(as in a finite automaton). Use for



Path Algorithms

Can produce generic algorithms, just need definitions for

Can take advantage of identities

Two kinds of algorithms

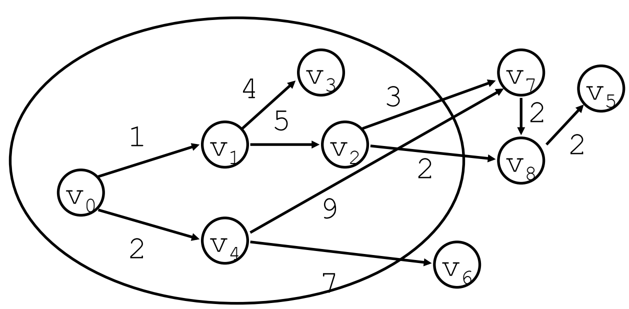
1.

2.

Can do 2. with

Dijkstra's Algorithm

Single-source, shortest path in a digraph



S—

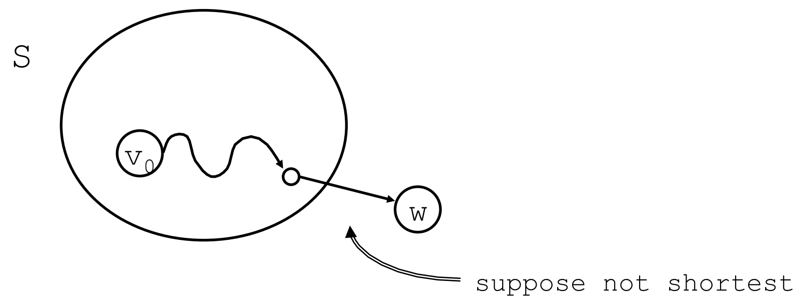
D—array of distances

General Step

Add node not in S with minimum D value to S.

Update distances to nodes in N-S

Is S still correct?



Full Algorithm

```
G = (N,E)
lab(v,w) - label
   represent with matrix
S ← {v0}
D[v0] ← 0
D[v] ← lab(v0,v) for rest
while S ≠ N do
   choose w
   S ←
   for all v in N-S do
      D[v] ←
```

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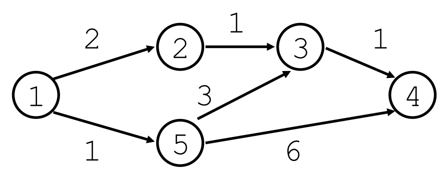
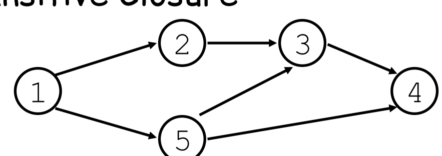
Example



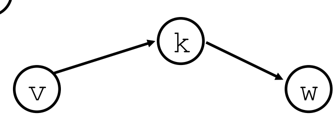
S	D[1]	D[2]	D[3]	D[4]	D[5]
{1}	0	2	∞	∞	1

Warshall's Algorithm

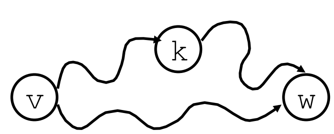
Transitive Closure



Basic op



Consider each `k` only once



Algorithm

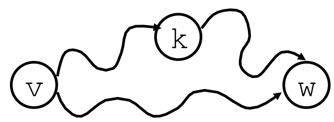
```
for k = 1 to n do
  for v = 1 to n do
    if a[v,k] then
      for w = 1 to n do
        if a[k,w] then
```


Example

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

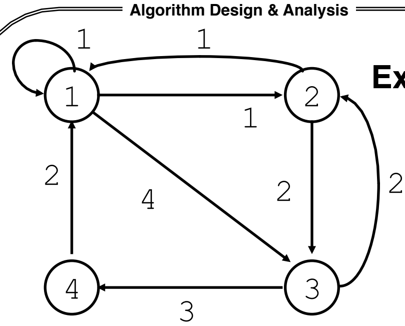
Floyd's Algorithm

All-pairs shortest path




```
for k = 1 to n do
  for v = 1 to n do
    if d[v,k] ≠ ∞ then
      for w = 1 to n do
        d[v,w] ←
```


Example



$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 1 & 4 & \infty \\ 1 & 0 & 2 & \infty \\ \infty & 2 & 0 & 3 \\ 2 & \infty & \infty & 0 \end{pmatrix} \end{array}$$

$d[i,j]^0$

Example 2

$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 1 & 4 & \infty \\ 1 & 0 & 2 & \infty \\ \infty & 2 & 0 & 3 \\ 2 & \infty & \infty & 0 \end{pmatrix} \end{array} \quad d[i,j]^0$$

$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} \quad & \quad & \quad & \quad \\ \quad & & & \\ \quad & & & \\ \quad & & & \end{pmatrix} \end{array} \quad d[i,j]^1$$

Example 3

$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 1 & 4 & \infty \\ 1 & 0 & 2 & \infty \\ \infty & 2 & 0 & 3 \\ 2 & 3 & 6 & 0 \end{pmatrix} \end{array} \quad d[i,j]^1$$

$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 1 & 3 & \infty \\ 1 & 0 & 2 & \infty \\ 3 & 2 & 0 & 3 \\ 2 & 3 & 5 & 0 \end{pmatrix} \end{array} \quad d[i,j]^2$$

Example 3

$d[i,j]^2$

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 3 & \infty \\ 2 & 1 & 0 & 2 & \infty \\ 3 & 3 & 2 & 0 & 3 \\ 4 & 2 & 3 & 5 & 0 \end{array}$$

$d[i,j]^3$

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & & & & \\ 2 & & & & \\ 3 & & & & \\ 4 & & & & \end{array}$$