CS 410/584, Algorithm Design & Analysis: Lecture Notes 3

Amortized Cost

- Not an algorithm design mechanism per se
- For analyzing cost of algorithms
  -
  -

Example: Exercise 17.3-6

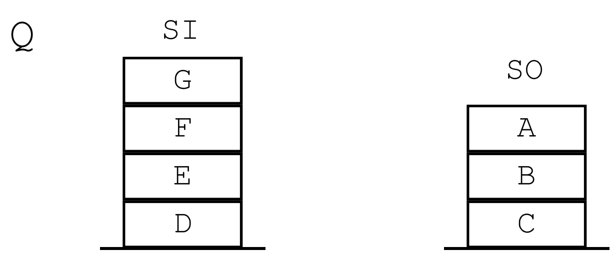
Implement a queue with two stacks

Why?

Queue Operations

```
enqueue(x, Q)


dequeue(Q)
  return(          )
  pop(SO)
```



A Complication

```
dequeue(Q)
  if empty(SO) then
    while not empty(SI) do


  return(top(SO))
  pop(SO)
```

Complexity

- operation `enqueue` is always
- in a sequence of `n` queue ops, one `dequeue` could be as bad as
- but total cost of sequence isn't so bad as

Consider: each element has $4

- pay $1 for
- pay $2 for
- pay $1 for

Union-Find

Assumptions

1.

2.

3.

4.

Naming by an element is not a limitation

Usually want to know if

Critical Assumption: the number of ops is proportional to

First Hack

Array `R[1..n]`

`R[i]` ≡ name of set

Initially, `R[i] =`

`Union(i,j)`: Scan `R`. For every element `k` with `R[k] = i`, set

i	1	2	3	4	5
R[i]	1	2	3	4	5

Each union costs

Improving Efficiency

More efficiency
- find just the elements of the set
- always choose

`R[i]` as before

If `j` is a "name", then `LIST[j]` points to

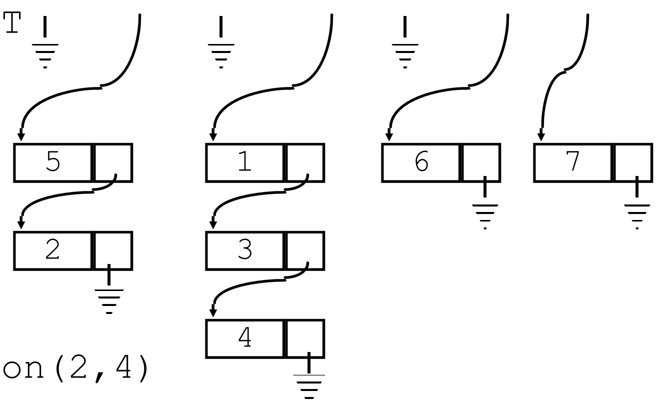
and `SIZE[j] =`

```
Union(i,j)  {assume i=    , j=    }
1. Assume SIZE[i]   SIZE[j]
2. Traverse LIST[i] and
3. Add LIST[i] to
4. Adjust SIZE[j]
```


Example

i	1	2	3	4	5	6	7
R[i]	4	2	4	4	2	6	7
SIZE	0	2	0	3	0	1	1

LIST



Union(2,4)

Time Complexity

Theorem: Can execute `n-1` `Union` operations in `O(n·log n)` time.

- Time for `Union` is proportional to

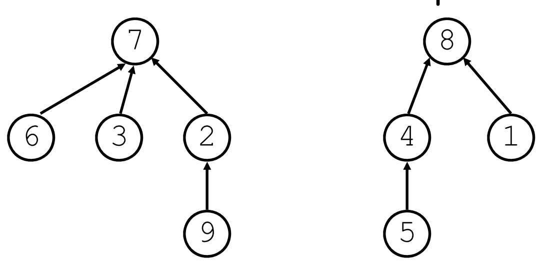
- Charge the cost to the elements moved when they go

- How many times may a single element be moved?

Tree-Structured Union-Find

Data structure: forest of trees, one per

Almost linear time for `n` ops



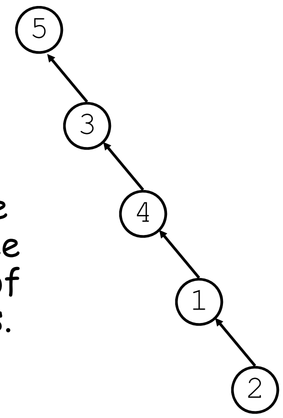
`Find(i) -`

`Union(i,j) -`

Balancing

Can get bad trees

Keep trees somewhat balanced:
always merge

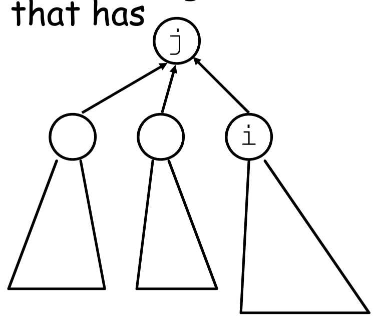


Lemma: If for every `Union`, the smaller tree is made a subtree of the larger, then any tree of height h has at least 2^h nodes.

Proof: If $h = 0$, then the tree must be

Proof, Continued

If $h > 0$, choose height h tree formed by `Union`'s that has

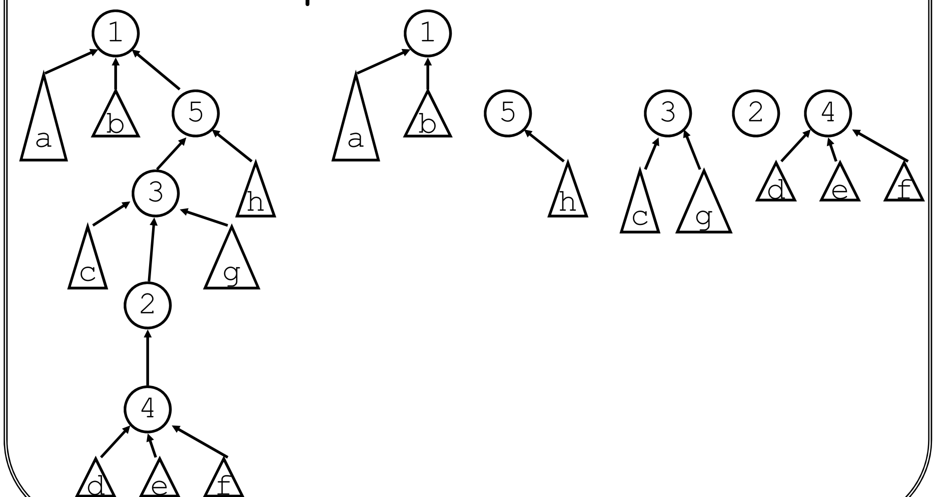


Look at last tree added, must have height

Before the addition, T_j has height

Path Compression

Further improvement on `Find` time



Complexity

Need funny functions to prove complexity

`F(0) =`

`F(i) =`

i	0	1	2	3	4	5
F(i)						

`G` is kind of an inverse

`G(n)` = the least `k` such that

In this universe, `G(n)` ≤ for all practical `n`

Complexity 2

Show that a sequence σ of `c·n Union/Find`'s can be done in

Assume: `Union` takes
`Find` takes "units" proportional to

Definition: The `u-height` (union height) of an element relative to

1. Let σ′ be σ with
2. Construct a forest using
3. `u-height(v)` is height of `v` in

Complexity 3

Lemma: There are at most

Proof: Node of `u-height = r` has at least descendents in forest `F`.

If `u-height(v) = u-height(w)`, sets of descendents are

There are at most distinct sets of 2^r elements, so

Corollary:

Complexity 4

Lemma: If during execution of σ (not), w is a proper descendent of v, then

Proof: w a proper descendent of v under σ implies

Put nodes into groups by u-height

u-height = r goes in

u-height	0-1	2	3-4	5-16	17-65536	65537-2^{65536}
group						

Complexity 5

How to count cost of Find's in $c \cdot n$ operations.

Split cost of Find between

1.

2. certain nodes

In the end have

sum over

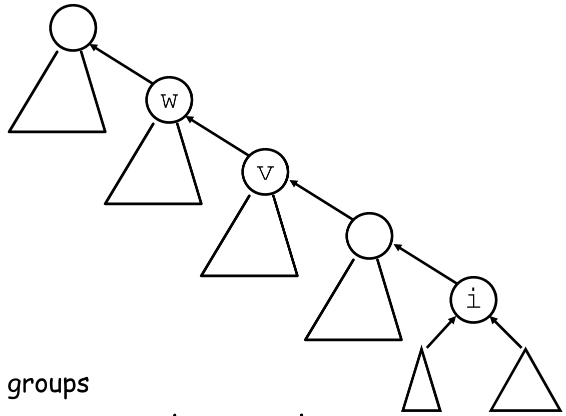
+

sum over

Suppose Find(i) traverses k nodes

- How to split up k "units"?

Complexity 6



Consider each node v on path

1. If
   - v is root
   - w is root
   - v, w in different groups
2. If v, w in same group and w not the root,

Complexity 7

Along any upward path, u-heights increase

At most different groups
so case 1 applies to at most

So a Find operation gets charged at most

Complexity 8

What happens when case 2 applies?

`v` gets a new parent `x`

and `u-height(x)` `u-height(w)`

- How many times can `v` get a new parent before its parent is in a higher group, and case 1 applies?

At most

Complexity 9

Charges to nodes:

Σ (max nodes in)(max moves for)

$$N(g) \leq \sum_{r=F(g-1)+1}^{F(g)}$$

max moves for a node in a group $\leq$

Graphs

An <u>undirected graph</u> `G` is a pair `(N, E)`

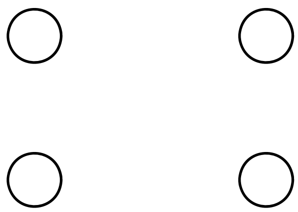
`N`

`E`

Example

`N = {a, b, c, d}`

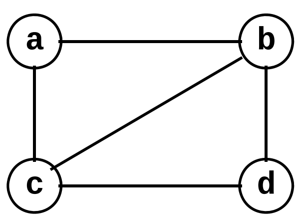
`E = {{a,b},{b,c},{a,c},{b,d},{c,d}}`



Paths

A <u>path</u> in graph `G = (N,E)`

A sequence $n_1, n_2, ..., n_k$ $k \geq 2$ of nodes such that



A path is <u>simple</u> if

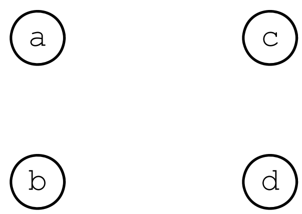
A path is a <u>cycle</u> if and $k \geq 3$

Directed Graphs

In a directed graph (digraph) `E` is

Example

`E = {(a,b),(b,c),(a,c),(b,d),`
`(c,d),(d,c)}`



Representation

Time and space complexity depends on how we capture the graph.

Edge list

`{a,b}, {b,c}, {a,c}, {b,d}, {c,d}`

Adjacency List

a →
b →
c →
d →

Representation 2

Adjacency matrix

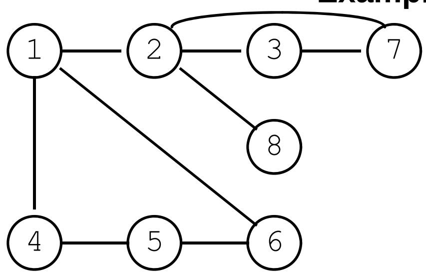
$$
\begin{array}{c}
\\
a \\ b \\ c \\ d
\end{array}
\begin{array}{c}
\begin{array}{cccc} a & b & c & d \end{array} \\
\begin{pmatrix}
1 & & & \\
1 & & & \\
1 & & & \\
0 & & &
\end{pmatrix}
\end{array}
$$

Depth-First Search–Undirected Graphs

A means to visit all nodes of an undirected graph

```
at node x
   pick edge {x,y}
   if y visited, pick another
    edge
   else
if no more edges, we're done
```

Example DFS



DFS Algorithm

```
DFS(v)
  mark v
  for each edge {v,w} do
    if w not marked then
```

"saved" edges form the <u>depth-first spanning tree</u>

T -

B -

DFS number

DFNum[v]

Time Complexity

<u>Theorem</u>: `DFS` requires `O(max(n,e))` steps on a graph with `n` nodes and `e` edges (given as an adjacency list).

Time spent looking for unmarked nodes

`DFS(v)` is called once for each node.

time spent, apart from recursive calls, is

$$\sum_{v \in N} (\#\text{nodes adjacent to } v)$$

Back Edges

If $\{v,w\} \in B$, then v is an ancestor of w in the spanning forest (or vice versa).

If v an ancestor of w in DFS tree, then `DFNum[v]` `DFNum[w]`.