Algorithm Design & Analysis

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Let's Solve a Problem

Longest Common Subsequence (LCS)

$S = a_1 a_2 \ . \ . \ . \ a_n$

A <u>subsequence</u> is

$a_{i_1} a_{i_2} \cdots a_{i_k} \qquad i_1 < i_2 < \cdots < i_k$

A B S Q U A T U L A T E

AQTUL

LATE

ε empty string

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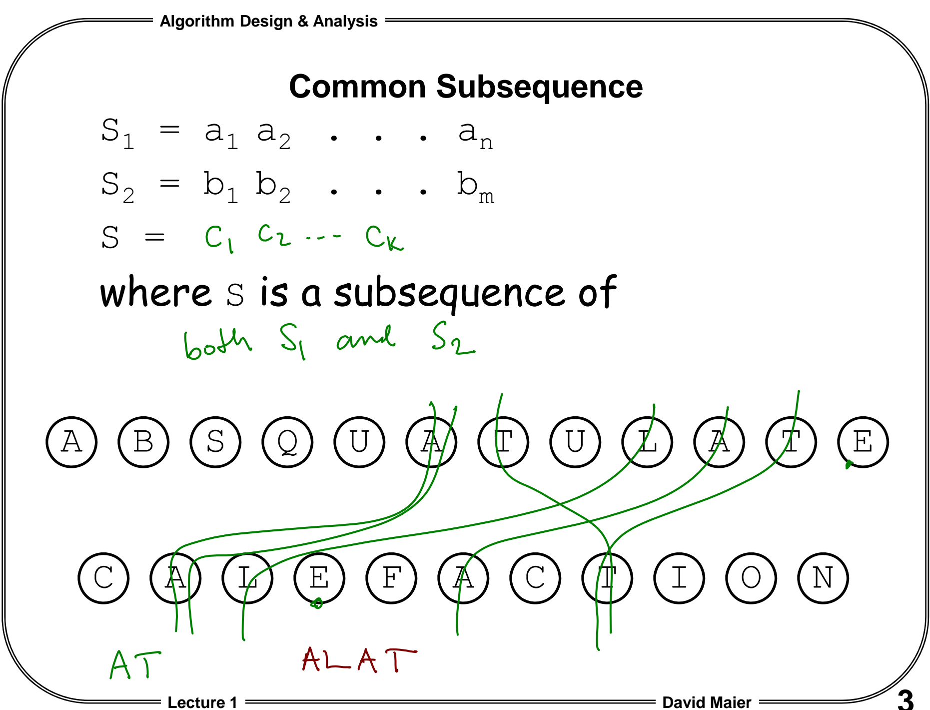

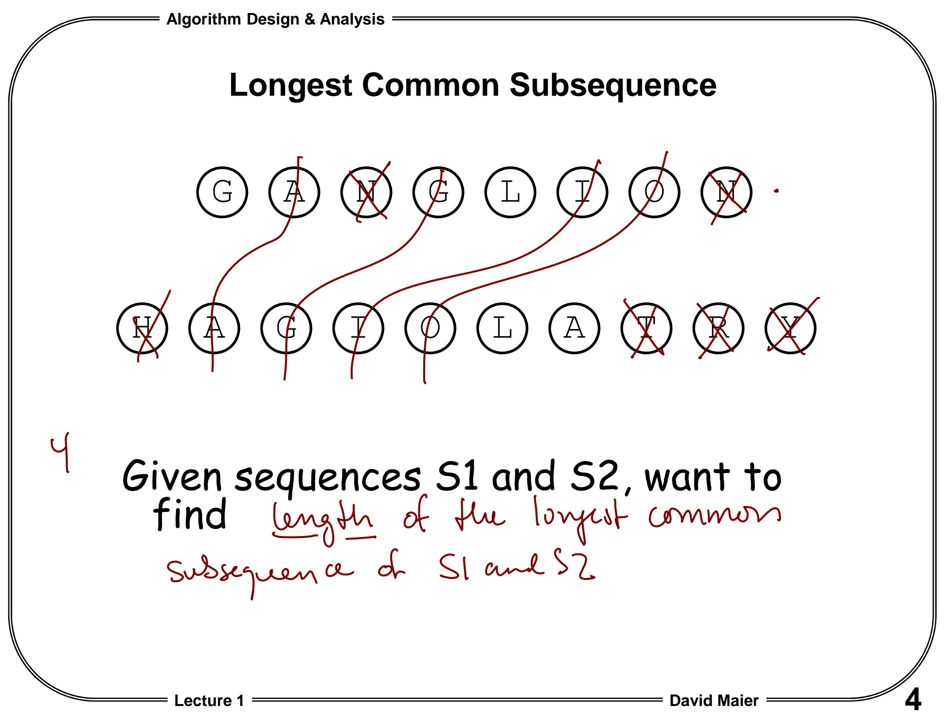

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Enumeration Algorithm

```
LCS(S1, S2)
max ← 0
for every subseq. R1 of S1
  for every subseq. R2 of S2
    if R1 = R2 and length(R1) > max
    then max ← length(R1)
return(max)
```

longest to shortest? quit at first one

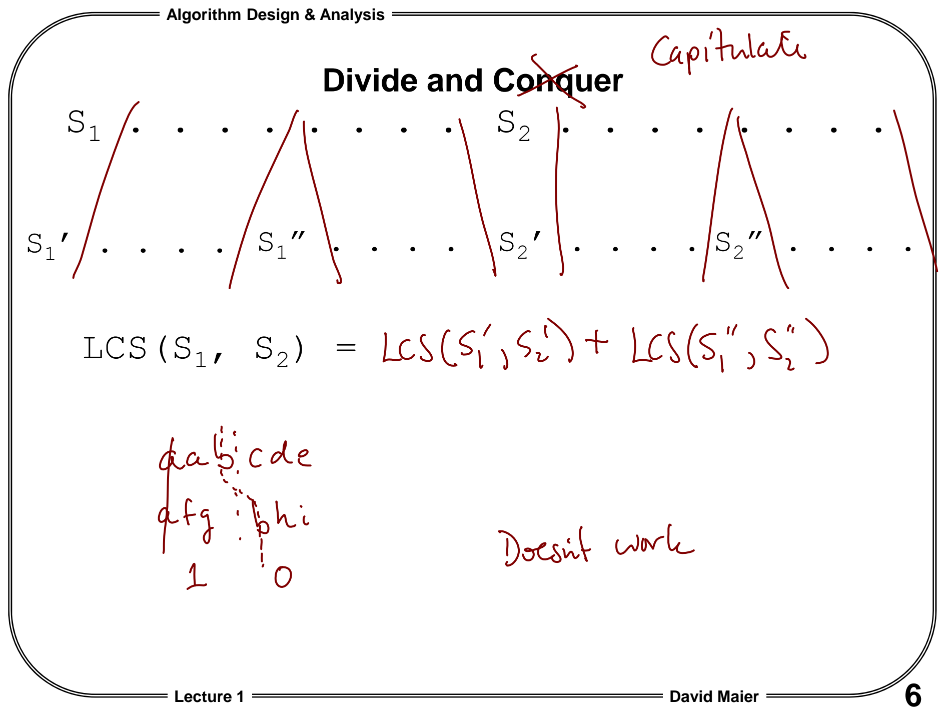
prune out letters in just one sequence

checks if R1 subseq of S2

$length(S_1) = n$ $length(S_2) = m$

time complexity = $2^n \cdot m$

comparisons of letters



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Recursion

LCS(4,5)

```
R   a  b  b  c  d  a
S   a  e  b  b  a  e  e
```

L(i,j) = length of LCS of R[1..i] S[1..j]

```
R[i] ≠ S[j]
   LCS(i,j)  = max(LCS(i-1,j ),LCS(i,j-1 ))
   LCS(6,7)  = max(LCS(5,7), LCS(6,6)) = 4
                      3 abb     4 abba

R[i] = S[j]
   LCS(i,j)  = LCS(i-1,j-1)  + 1
   LCS(6,5)  = LCS(5,4) + 1
                 3  +  1  =  4
```

LCS(0,j) = LCS(i,0) = 0

Recursion 2

R

S

```
LCS(i,j)
   if i=0 or j=0 then LCS ← 0
   else if R[i] ≠ S[j] then
      LCS ← max(LCS(i-1,j),LCS(i,j-1))
   else  (R[i] = S[j])
      LCS ← LCS(i-1,j-1) +1
```

Count comparisons

worst case

2 twice as many of the subproblem

c(LCS(i,j)) = 1 + c(LCS(i-1,j)) + c(LCS(i,j-1))

2^n behavior

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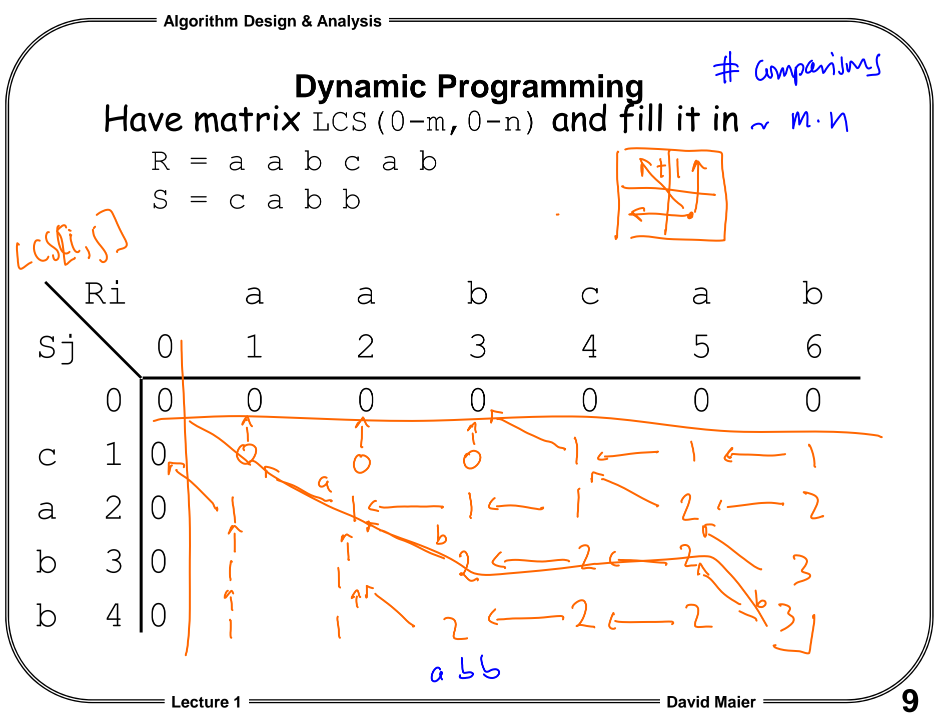

Algorithm Longest

```
1. Set LCS[i,j] ← 0 if i=0 or j=0
2. for j = 1 to n do
     for i = 1 to m do
       if R[i] ≠ S[j] then
       LCS[i,j]
         ← max(LCS[i-1,j],LCS[i,j-1])
       else
         LCS[i,j] ← LCS[i-1,j-1]+1
3. return LCS[m , n ]
```

Time complexity m.n comparisons

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Logistics

Course web page is
http://www.cs.pdx.edu/~maier/cs584

Expect class email list to be
cs584@cs.pdx.edu
(Watch the web page for sure.)

Will try to post weekly lecture slides by the evening before Monday class

Class Activity and Quiz

Work in groups on a sample execution of the Freq algorithm

Will have a short quiz on it –

The quiz doesn't count, but should help you assess if you are prepared for this class

I assume you know the material from CS310

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Study of Algorithms

- Design techniques

  divide + conquer, dynamic programming
- Specific algorithms

  building blocks geometry
- Proofs of correctness
- Complexity analysis
  - measure — codesize
  - time it takes, space it consumes

    as a function of input size
  - worst case
  - average case

Study of Problems

- What's the best any algorithm can do?

  lower bound – usually hard

  upper bound – exhibiting an algorithm
- Relatedness – use algorithms for one problem to solve another

  transitive closure + boolean matrix mult.
- Intractability of problems

  There's no efficient algorithm

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Complexity of an Algorithm

Amount of time and space required as a function of size of input n

T(n) = maximum time required over all inputs of size n

S(n) = maximum space required over all inputs of size n

Input size
- # of elements
- # of bits

Time measures
- count instructions
- count a particular op e.g. comparisons

Time Complexity of a Problem

Minimum time over all algorithms

to get

upper bound: exhibit alg.

lower bound: ad hoc argument; reduction from another problem

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Asymptotic Complexity

order notation

f(n) has order O(g(n)) if there is a c > 0

f(n) ≤ c · g(n) for all n > n_0 for some n_0

f(n) = n^2 + 3n + 4

f(n) has $O(n^2)$

f(n) ≤ 2 · g(n) for n ≥ 5

c ← (the 2); n_0 ← (the 5)

Asymptotic Notation

O(g(n)) = { f(n) there are positive c and n_0 such that f(n) ≤ c · g(n) when n > n_0 }

n^2 + 3n + 4 = O(n^2)

means

$n^2 + 3n + 4 \in O(n^2)$ $n^2 + 3n + 4 \in O(n^3)$

Theta Θ(g(n)) = {f(n) | there are positive c_1, c_2 and n_0 such that $c_1 g(n) \le f(n) \le c_2 g(n)$, $n > n_0$}

n^2 + 3n + 4 $\in O(n^3)$

$\notin \Theta(n^3)$

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Asymptotic Notation 2

Note

$n^2 + 3n + 4 \in O(n^3)$

$n^2 + 3n + 4 \notin \Theta(n^3)$

$\Theta(n^3) \subseteq O(n^3)$

Omega $\Omega(g(n)) = \{f(n) \mid$ there are positive c and n_0 such that $c \cdot g(n) \le f(n)$ $n > n_0$ $\}$

asymptotic

- upper bound O
- tight bound Θ
- lower bound Ω

Asymptotic Notation 3

Theorem

$f(n) \in O(g(n))$ and
$f(n) \in \Omega(g(n))$
imply
$f(n) \in \Theta(g(n))$

Other facts

$f(n) \in \Theta(h(n))$ and $h(n) \in \Theta(g(n))$
imply $f(n) \in \Theta(g(n))$

$f(n) \in \Theta(g(n))$
implies
$g(n) \in \Theta(f(n))$

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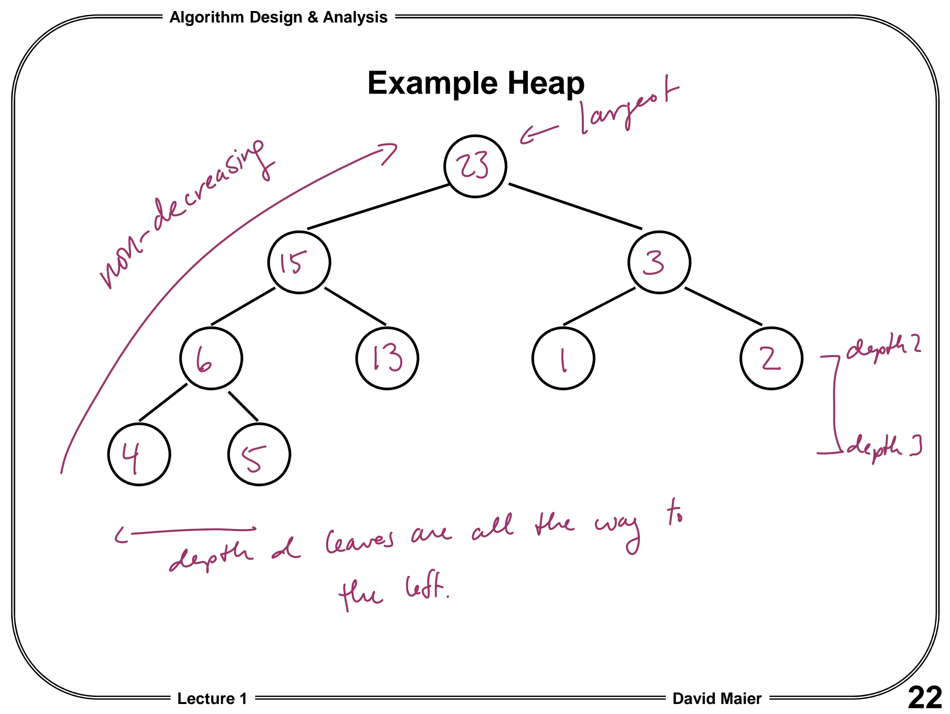
Heapsort

Heap: binary tree where all leaves at depth d-1 or d

node labels are items to sort

value at node ≥ all values in subtrees



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Heapsort Algorithm

Heapsort (make heap)

1. remove label of root
2. move label of some leaf to root
3. delete that leaf
4. remake heap

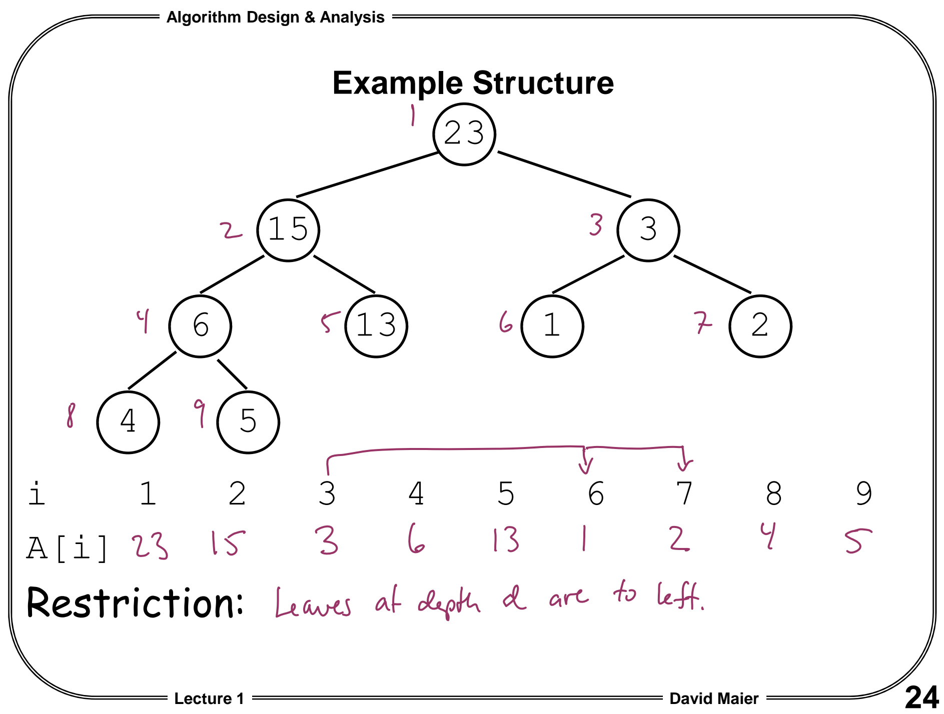
Implicit data structure

A[1..n] — array of items to sort

A[1] is root

left and right children of A[i] are at

A[2i] A[2i+1]



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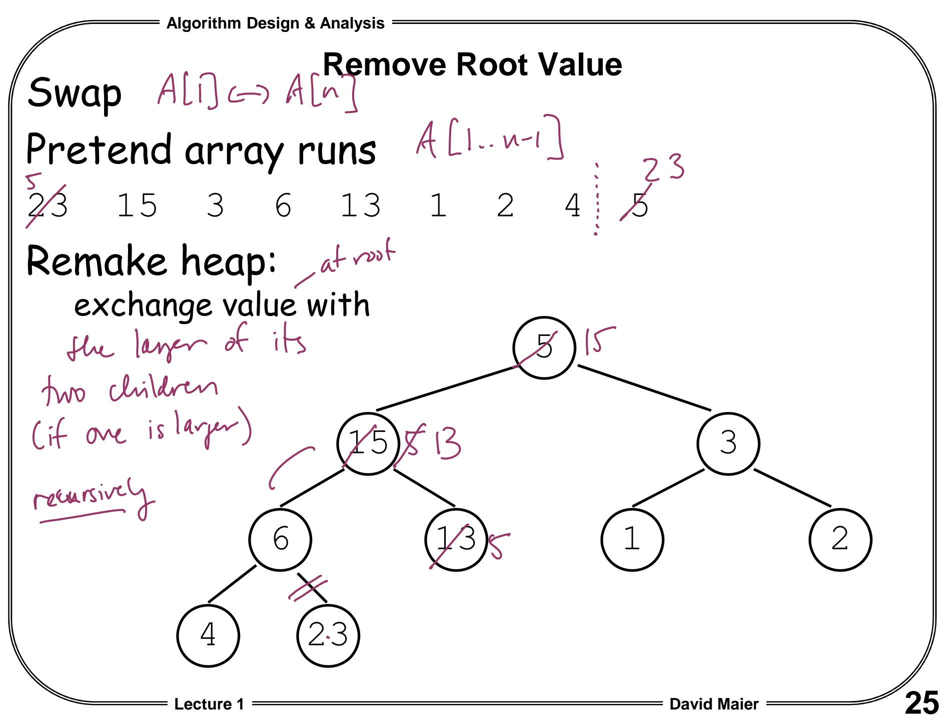

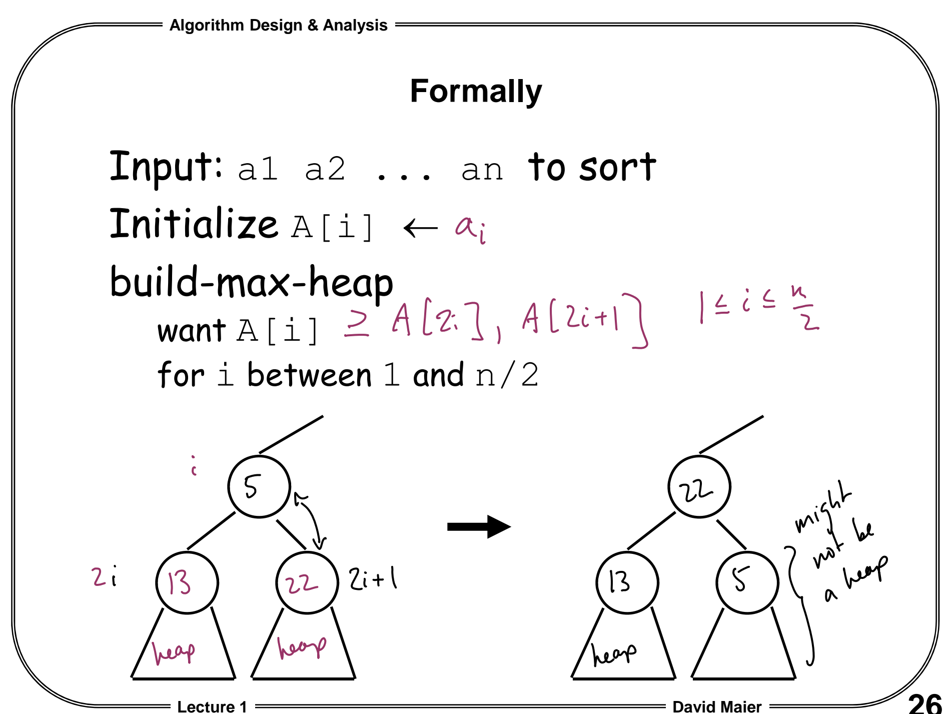

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Heap Construction

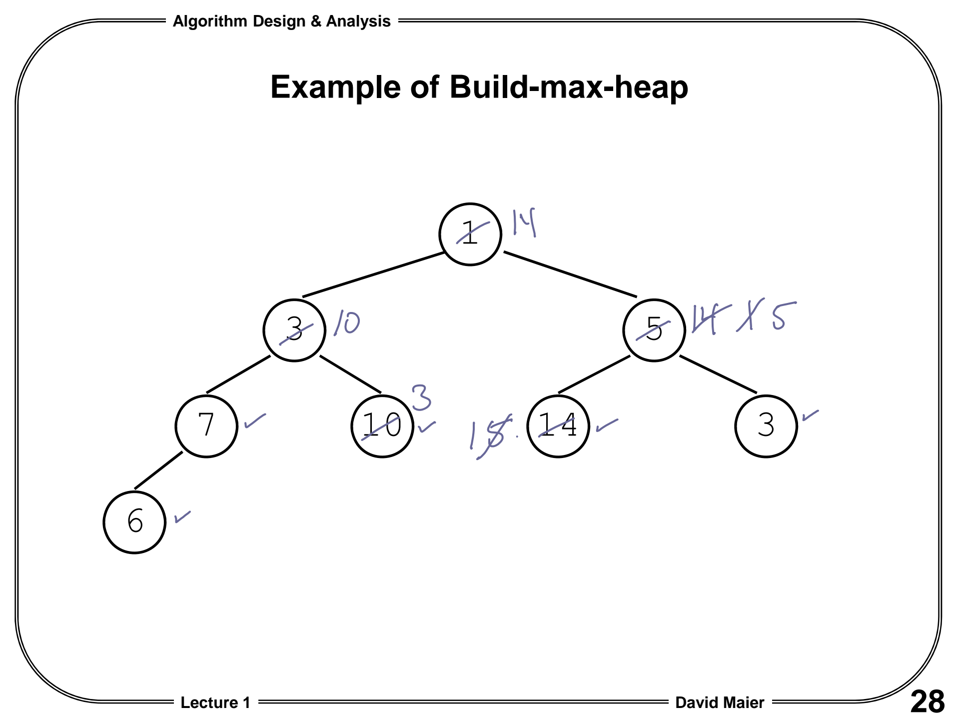
Assume: A[i+1] ... A[j] have heap property

Want: A[i] to be root of heap

```
max-heapify(i,j)        -> uses j
   if i not a leaf and
     a child of i has a greater value
   then
      let k be child with larger value
      A[i] <-> A[k]
      max-heapify(k,j)

build-max-heap
   for i = n/2 downto 1 do
      max-heapify(i,n)
```


Example of Build-max-heap



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Complexity of Build-max-heap

Theorem: build-max-heap has `O(n)` time complexity

Define `T(h)` as the time to max-heapify a node of height `h` *(distance above a leaf)*

`T(h)` ≤ $T(h-1) + c$

`T(h)` $\le h \cdot c \in O(h)$

Build-max-heap calls max-heapify once per node.

How many nodes of height `i`?

at most $\frac{n}{2^{i+1}}$

(log n = max height)

$cn \sum_{i=1}^{\log n} i \cdot \frac{1}{2^{i+1}}$ — $\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \frac{5}{32} \ldots$ *(limit of series)* $c \cdot n \cdot k = O(n)$

Heapsort Algorithm

```
heapsort
    build-max-heap
    for i = n to 2 do
```

O(n) - build-max-heap

A[1] ↔ A[i]
max-heapify (1, i-1) *(pretending the array runs from 1 to i-1)*

Sorted list ends up in `A[1] ... A[n]`

Time complexity: *max-heapify (1, i)* $O(\lg i)$

$\log n + \log n-1 + \log n-2 + \cdots + \log 2 + \log 1 \in O(n \lg n)$

 # comparisons is $O(n \lg n)$

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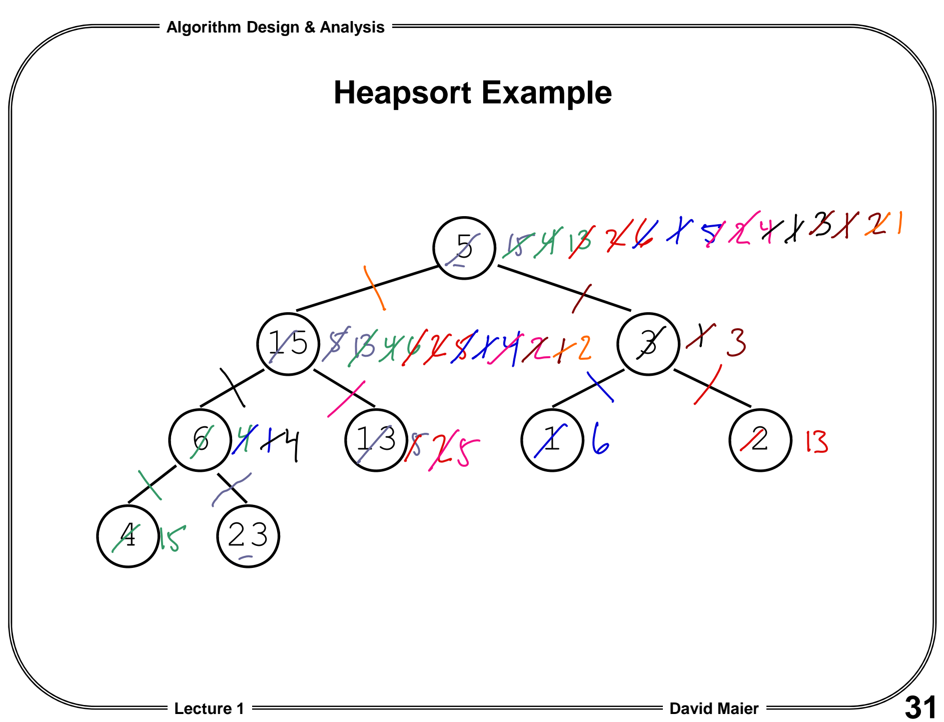

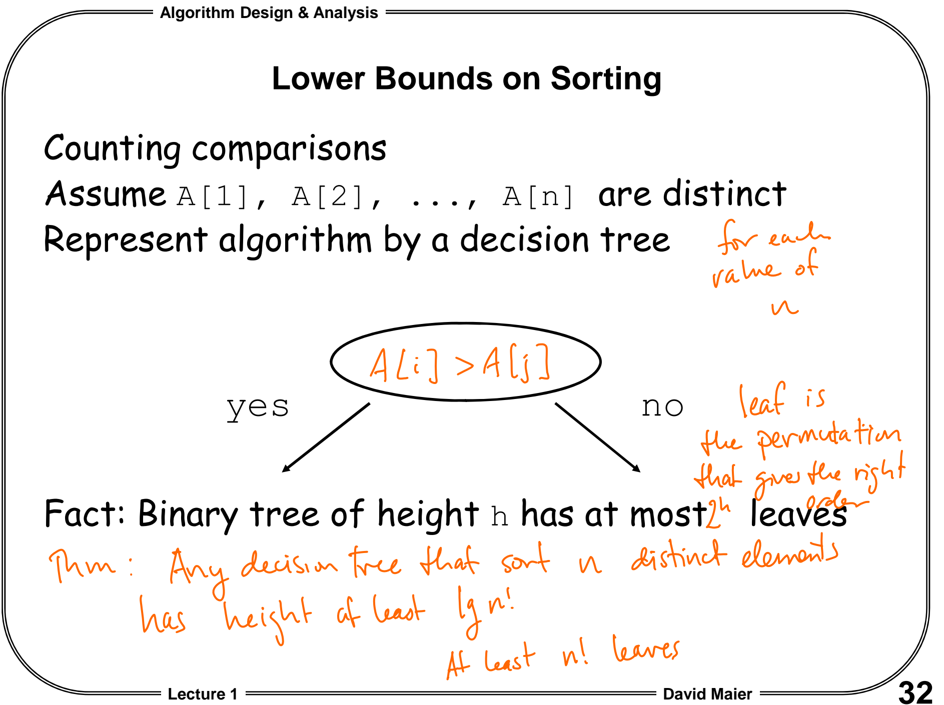

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Consequence

Corollary: Any algorithm for sorting by comparisons needs $cn \lg n$ compares for some $c > 0$ and large n.

$$n! \geq n(n-1) \ldots (n/2) \geq (n/2)^{n/2}$$

$$\lg n! \geq \lg (n/2)^{n/2} = \frac{n}{2} \log \frac{n}{2} = \frac{n}{2}(\log n - \log 2)$$

$$\geq \frac{1}{4} n \lg n \quad \text{if } n \geq 4$$

$n \log n \geq \log n!$ since

$$n^n \geq n!$$

Dynamic Programming

Two elements Bellman's principle

Correctness
- Optimal substructure: solution contains optimal solutions to subproblems

  LCS(7,7) (as sequence) contains LCS(6,6) if $a_7 = b_7$

Utility
- Overlapping subproblems: if done recursively

  LCS(6,5) used for LCS(7,5), LCS(6,6), LCS(7,6)

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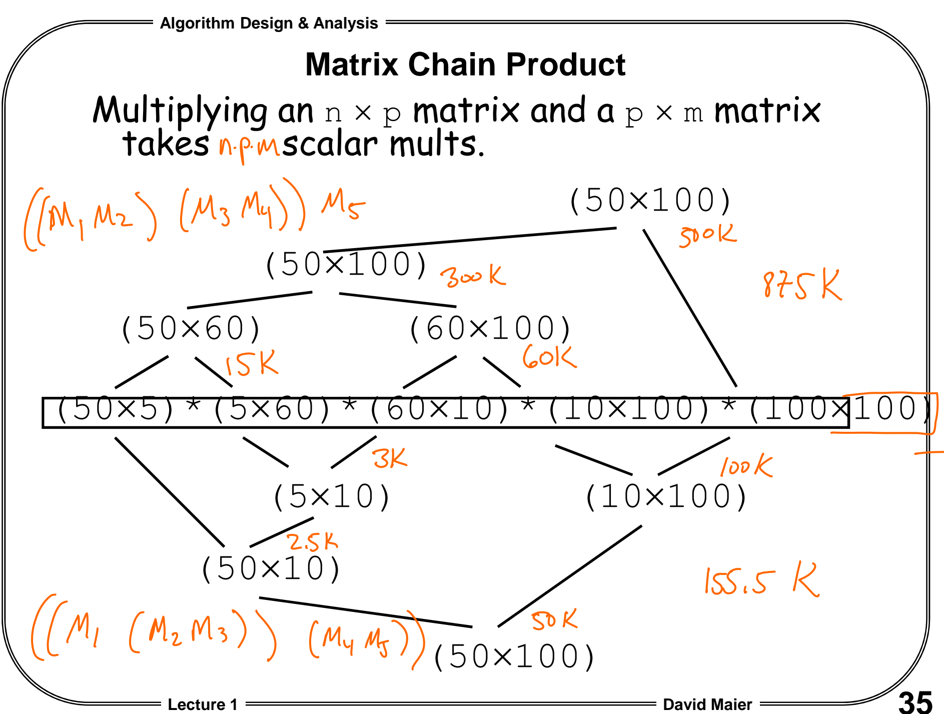

Algorithm Design & Analysis

Recursive Approach

Have matrices $M_1*M_2*\ ...\ *M_n$

Let Mult[i,j] be cost to multiply $M_i * \cdots * M_j$

Let $p_0, p_1, ..., p_n$ be dimensions such that M_i has dimension M_i has dims $p_{i-1} \times p_i$

Assume Mult[i,i] = 0

Then Mult[i,j] =

$$\min_{i \le k < j} (\text{Mult}[\underbrace{i, k}_{p_{i-1} \times p_k}] + \text{Mult}[\underbrace{k+1, j}_{p_k \times p_j}] + p_{i-1} p_k p_j)$$

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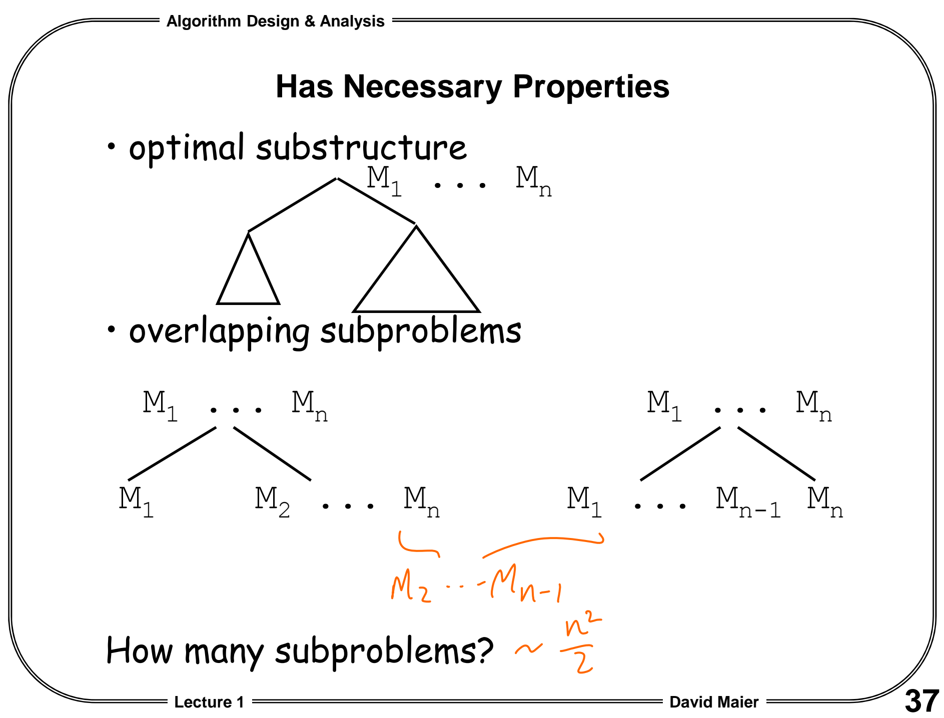

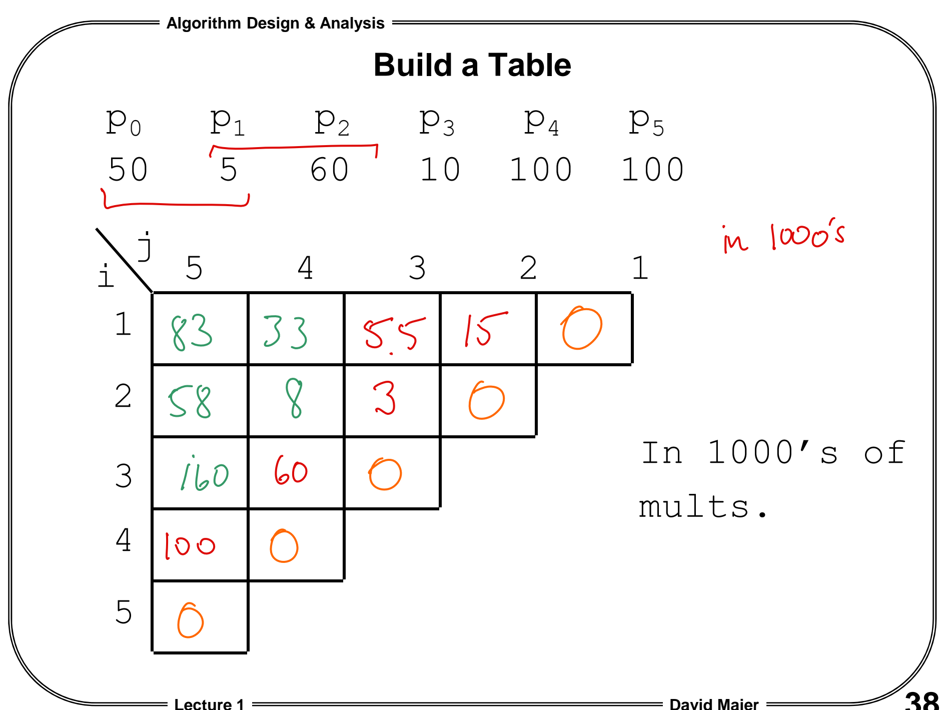

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Mult[i,j] Values

```
1,3     1,1 + 2,3 + 50*5*10
        1,2 + 3,3 + 50*60*10

2,4     2,2 + 3,4 + 5*60*100
        2,3 + 4,4 + 5*10*100

3,5
```


Mult[i,j] Values 2

```
1,4     1,1 + 2,4 + 50*   *100
        1,2 + 3,4 + 50*   *100
        1,3 + 4,4 + 50*   *100

2,5     2,2 + 3,5 + 5*   *100
        2,3 + 4,5 + 5*   *100
        2,4 + 5,5 + 5*   *100
```


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Algorithm Design & Analysis

Mult[i,j] Values 3

```
1,5   1,1 + 2,5 + 50* 5 *100
      1,2 + 3,5 + 50* 60 *100
      1,3 + 4,5 + 50* 10 *100
      1,4 + 5,5 + 50* 100 *100
```

Handwritten annotations: 0, 58, 25 (= 83 ✓); 1.5, 160, 300; 5.5, 100, 50; 33, 0, 500

Complexity

$O(n^3)$

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