Planning Approximations to the average length of vehicle routing problems with time window constraints

Miguel Andres Figliozzi[1]

ABSTRACT

This paper studies approximations to the average length of Vehicle Routing Problems (VRP) with time window, route duration, and capacity constraints. The approximations are valuable for the strategic and planning analysis of transportation and logistics problems. Using asymptotic properties of vehicle routing problems and the average probability of successfully sequencing a customer with time windows a new expression to estimate VRP distances is developed. The increase in the number of routes when time constraints are added is modeled probabilistically. This paper introduces the concept of the average probability of successfully sequencing a customer with time windows. It is proven that this average probability is a unique characteristic of a vehicle routing problem. The approximation proposed is tested in instances with different customer spatial distributions, depot locations and number of customers. Regression results indicate that the proposed approximation is not only intuitive but also predicts the average length of VRP problems with a high level of accuracy.

KEYWORDS: Vehicle Routing Problem, Distance Estimation, Time Window Constraints.

[1] Assistant Professor, College of Engineering and Computer Science, Department of Civil Engineering, Portland State University, Portland, USA, e-mail: figliozzi@pdx.edu

1. INTRODUCTION

In many logistics problems it is necessary to estimate the distance that a fleet of vehicles travel to meet a set of customer demands. Traveled distance is not only an important element of carriers' variable costs but it is also a key input in tactical and strategic models to solve problems such as facility location, fleet sizing, and network design.

The transportation decisions associated with high value - high time sensitive products, are the most demanding activities in terms of transport service requirements and usually require service within a hard time windows (Figliozzi, 2006). Time windows are a key constraint also for make to order-JIT production systems as well as emergency repair work and express (courier) delivery services. Time windows have a significant impact on decreasing the efficiency of routes, reducing service areas, and significantly increasing distance travelled (Figliozzi, 2007).

Despite the growing implementation of customer-responsive and made-to-order supply chains, the impact of time window constraints and customer demand levels on average VRP[2] distance traveled has not yet been studied in the literature. The existing body of literature has mostly focused on the estimation of distances for either the Traveling Salesman Problems (TSP) or the capacitated vehicle routing problems (CVRP). This research provides an intuitive and parsimonious mathematical framework to estimate average distances in VRP problems.

The paper is organized as follows: Section two provides a literature review. Section three presents asymptotic results for the VRP and expressions to estimate the additional number of routes due to time window constraints. Section four presents a new expression to estimate distance traveled. Section five describes the experimental design and test results. Section six ends with conclusions.

2. LITERATURE REVIEW

A seminal contribution to the estimation of the length of a shortest closed path or tour through a set of points was established by Beardwood et al. (1959) . These authors demonstrated that for a set of n points distributed in an area A, the length of the TSP tour through the whole set asymptotically converges, with a probability of one, to the product of a constant k by the square root of the number of points and the area, i.e. $k\sqrt{nA}$ when $n \to \infty$.

The asymptotic validity of this formula for TSP problems was experimentally tested by Ong and Huang (1989) using a nearest neighbor and exchange improvement heuristics. With

[2] Hereafter, VRP denotes vehicle routing problems with time window, route duration, and capacity constraints.

an Euclidian metric and a uniform distribution of customers the constant term has been estimated at $k = 0.765$ (Stein, 1978). For reasonably compact and convex areas, the limit provided by Beardwood et al. converges rapidly (Larson and Odoni, 1981). Jaillet (1988) estimated the constant $k = 0.97$ for a Manhattan metric.

Approximations to the length of capacitated vehicle routing problems were first published in the late 1960's and early 1970's (Webb, 1968, Christofides and Eilon, 1969, Eilon et al., 1971). Webb (1968) studied the correlation between route distance and customer-depot distances. Eilon et al. (1971) proposed several approximations to the length of CVRP based on the shape and area of delivery, the average distance between customers and the depot, the capacity of the vehicle in terms of the number of customers that can be served per vehicle, and the area of a rectangular delivery region.

Daganzo (1984) proposed a simple and intuitive formula for the CVRP length when the depot is not necessarily located in the area that contains the customers.

$$CVRP(n) \approx 2rn/Q + 0.57\sqrt{nA}$$

In this expression $CVRP(n)$ is the total distance of the CVRP problem serving n customers, the average distance between the customers and the depot is r, and the maximum number of customers that can be served per vehicle is Q. Hence, the number of routes m is *a priori* known and can be calculated as n/Q. Daganzo's approximation can be interpreted as having: (a) a term related to the distance between the depot and customers and (b) a term related to the distance between customers. The coefficients of Daganzo's approximation were derived assuming $Q > 6$ and $n > 4Q^2$. Daganzo's approximation works better in elongated areas as the routes were formed following the "strip" strategy. Robuste et al. (2004) tested Daganzo's approximation using simulations and elliptical areas; they propose adjustments based on area shape, vehicle capacity, and number of customers. A dissertation produced by Erera (2000) extended the usage of continuous approximations to estimate the distance of detours and routes in a stochastic version of the CVRP.

Chien (1992) carried out simulations and linear regressions to test the accuracy of different models to estimate the length of TSP. Chien tested rectangular areas with 8 different length/width ratios ranging from 1 to 8 and circular sectors with 8 different central angles ranging from 45 to 360 degrees. Exact solutions to solve the TSP problems were used and the size of the problems ranged from 5 to 30 customers. The depot was always located at the origin, i.e. the left-lower corner of a rectangular area. Chien randomly generated test problems and using linear regressions found the best fitting parameters. The mean absolute percentage error (*MAPE*) was the benchmark to compare specifications. Chien finds that the lowest *MAPE* for the best model is equal to 6.9%.

$$TSP(n) \approx 2.1r + 0.67\sqrt{nR} \qquad R^2 = 0.99 \qquad MAPE = 6.9$$

In this expression, $TSP(n)$ is the total distance of the TSP problem serving n customers. The area of the smallest rectangle that covers the customers is denoted R. The usage of R instead of the total area, A, containing all customers may not be convenient for planning purposes when there may be many possible subsets of customers that are not known *a priori*. Chien also estimated the previous models for each of the 16 different regions; linear regression R^2 and $MAPE$ are reported for each type of region and model. The estimated parameters change according to the shape of the region.

Kwon et al. (1995) also carried out simulations and linear regressions but in addition to Chien's work also used neural networks to identify better approximations. To test the accuracy of different models they tested TSP problems in rectangular areas with 8 length/width ratios ranging from 1 to 8. Models were estimated with the depot being located at the origin and at the middle of the rectangle. The sizes of the problems range from 10 to 80 customers. Kwon et al. proposed approximations that make use of the geometric information proportioned by the ratio length/width of the rectangle and a shape factor S. The results obtained for the depot located at the origin are as follows:

$$TSP(n) \approx [0.83 - 0.0011(n+1) + 1.11S/(n+1)]\sqrt{nA} \qquad R^2 = 0.99 \qquad MAPE = 3.71$$

$$TSP(n) \approx 0.41r + [0.77 - 0.0008(n+1) + 0.90S/(n+1)]\sqrt{nA} \qquad R^2 = 0.99 \qquad MAPE = 3.61.$$

By accounting for the shape of the area, Kwon et al. improved the accuracy of the estimations although this came at the expense of adding two extra terms. R^d is defined as the area of the smallest rectangle that covers the customers and the depot; with the depot located at the center of the rectangle the results obtained by Kwon et al. are as follows:

$$TSP(n) \approx [0.87 - 0.0016(n+1) + 1.34S/(n+1)]\sqrt{nR^d} \qquad R^2 = 0.99 \qquad MAPE = 3.88$$

$$TSP(n) \approx 1.15r + [0.79 - 0.0012(n+1) + 0.97S/(n+1)]\sqrt{nR^d} \qquad R^2 = 0.99 \qquad MAPE = 3.70$$

It can be observed that *MAPE* slightly increases when the depot is located at the center of the rectangle. Kwon et al. also used neural networks to find a model that better predicts TSP length. They concluded that the capability of neural networks to find "hidden" relationships provides a slight advantage against regression models. However, the models are less parsimonious and the approximations harder to interpret in geometric terms.

A simple and intuitive analytical model of VRP with time window constraints is provided by Daganzo (1987a, 1987b). Daganzo divides a day into time periods or bins of equal length and then clusters customers in rectangles. Each customer is then placed in a balanced time period or bin, consistent with his or her time window; this allows a simplification of the problem as customer individual time window characteristics are now associated with a time period. Using this time bin-cluster first-route second approach, Daganzo analyzes main routing tradeoffs and determines that distance traveled is a function of the square root of the number of time periods and that lower distances are possible when routes are allowed to overlap. Different approximations are provided if the dominant constraint is either vehicle capacity or route duration. Although Daganzo's formulas are useful and intuitive they are not easily applied to estimate VRP distance since his approach does not guarantee feasibility. Unfortunately, no systematic method or general expression for clustering and determining the number of periods that guarantees balanced periods and feasible routes is provided.

Approximations to the average length of vehicle routing problems have recently been contributed by Figliozzi (2008b) to estimate VRP distance when the number of customers served (n) and the number of routes (m) are given. The formula proposed accounts for the tradeoffs between connecting distance and local tour distance as the number of routes increases:

$$VRP(n) \approx k_l \frac{n-m}{n}\sqrt{An} + m2r \,.$$

The term $(n-m)/n$ is shown to improve *MAPE* values in problems with capacity constraints, time windows constraints, and a varying number of customers served (n).

3. CHARACTERIZATING THE IMPACT OF TIME WINDOW CONSTRAINTS

This section introduces a probabilistic approach to capture the impact of time windows on distance traveled for VRP instances that serve $N = \{1, 2, ..., n\}$ customers. Associated with each customer $i \in N$ there is a quintuplet $(x_i, q_i, s_i, e_i, l_i)$ that represents, respectively, the coordinates, demand, service time, earliest service starting time, and latest service ending time. The depot quintuplet is denoted $(x_0, q_0, s_0, e_0, l_0)$ with $q_0 = 0, s_0 = 0$ and $e_0 = 0$. The distance between each customer $i \in N$ and the depot is denoted $d(x_i)$; feasibility conditions include $d(x_i) \le l_i$, $2d(x_i) + s_i \le l_0$, and $q_i \le Q$. Customers with time windows are drawn from a probability measure ν with bounded support. Without loss of generality, attributes of the quintuplet are scaled and shifted so they belong to the real interval $[0,1]$. The coordinates x_i

are independently and identically distributed according to a distribution with compact support in $\Re^2$, $[0,1]\times[0,1]$; the customer parameters (q_i, s_i, e_i, l_i) are drawn from a joint probability distribution Φ with a continuous density function ϕ. The support of ϕ is the feasible subset of $(x_1, x_2, x_3, x_4) \in [0,1]^4$. It is also assumed that costumer locations and their parameters are independent of each other.

Customers without time windows are drawn from the same probability measure but their time windows are relaxed, i.e. (e_i, l_i) is replaced by (e_0, l_0). The "relaxed" probability measure is denoted μ, whose support is the feasible subset of $(x_1, x_2) \in [0,1]^2$ with $x_3 = 0, x_4 = 1$. The expected number of routes needed to serve n customers with and without time windows is denoted $m_\nu(n)$ and $m_\mu(n)$ respectively.

Known results for the capacitated vehicle routing problem (Bramel et al., 1992) indicate that:

$$\lim_{n\to\infty} \frac{CVRP^*(n,\mu)}{n} = 2\gamma_\mu\, E(d)$$

where $\gamma_\mu > 0$ is a constant that depend only on μ, $E(d)$ is the expected distance between the depot and customers, and $CVRP^*(n)$ is the best VRP solution for travel distance. The ratio $1/\gamma_\mu$ is the average number of customers per route. Similar results can be derived for the vehicle routing problem with time windows (Bramel and Simchi-Levi, 1996, Federgruen and Van Ryzin, 1997):

$$\lim_{n\to\infty} \frac{VRP^*(n,\nu)}{n} = 2\gamma_\nu\, E(d)\,.$$

The next lemma provides a useful bound for the additional number of routes due to time window constraints.

Lemma 1. The contribution of time windows to the distance traveled is bounded. Asymptotically, the number of additional routes due to time window constraints can be expressed as kn, being k a constant such that $0 \le k \le 1$.

Proof. Asymptotically, the contribution of time windows to the distance travelled per customer can be expressed as $2\,n\,E(d)(\gamma_\nu - \gamma_\mu)$. The increase in the number of routes due to time windows, denoted $m_{\nu\mu}$, can be approximated by $m_{\nu\mu}(n) = n(\gamma_\nu - \gamma_\mu) = m_\nu(n) - m_\mu(n)$. There cannot be more routes than customers, hence $1 \ge \gamma_\nu$. Time windows, additional constraints, cannot reduce the VRP distance; hence $(\gamma_\nu - \gamma_\mu) \ge 0$.

The increase in the number of routes when time constraints are added is modeled probabilistically. Given any two customers $i, j \in N$ there is a probability p_{ij} that a vehicle can successfully visit customer j after visiting customer i without violating $j's$ time window. In general, p_{ij} is a random variable that will depend on the probability measure ν. The goal is to find an expression that provides the average number of additional vehicles needed due to time window constraints, i.e. $m_{\nu\mu}(n)$. An exact solution using p_{ij} is likely to be intractable and to the best of the author's knowledge there is no general analytical expression that can be used to estimate the impact of time window constraints on VRP distances.

To model $m_{\nu\mu}(n)$, the concept of an average probability of successfully sequencing any given customer with time window constraints is introduced; let's denote this average success probability p_ν. Let's denote $b = 1/\gamma_\mu$ as the average number of customers per route or "bin" without time window constraints. The probability associated to finding a feasible route with $c \le b$ customers, each with time window constraints, can be expressed as:

$$p(c) = (p_\nu)^{c-1}$$

By definition $p(1) = 1$ because it is assumed that all customers can feasibly be served from the depot. When $b = 1$ the number of routes is simply $m = n$. When $b = 2$, the number of expected routes needed to serve n customers can be expressed as the weighted sum of routes with one and two customers:

$$\frac{n}{2} p(2) + n(1 - p(2))$$

and generalizing for any b:

$$E[m_\nu(n)] = \sum_{c=1}^{c=b} \frac{n\,p(c)}{c} \prod_{j=c+1}^{b} (1 - p(j)) \ .$$

A similar expression can be found in the work of Diana et al. (2006) which estimated demand responsive transit fleet sizes. The expected number of additional routes due to time window constraints, $E[m_{\nu\mu}]$, can be expressed as:

$$E[\,m_{\nu\mu}(n)] = [\sum_{c=1}^{c=b} \frac{n\,p(c)}{c} \prod_{j=c+1}^{b} (1 - p(j))\,] - \frac{n}{b} \qquad (1)$$

Lemma 2. The expected number of additional routes due to time window constraints, $E[m_{\nu\mu}(n)]$, is a continuously decreasing function of p_ν.

Proof. The complete proof is presented in Appendix A; a sketch of the proof is presented in this section. The sum of weight factors $w(c)$

$$w(c) = p(c) \prod_{j=c+1}^{b} (1 - p(j))$$

adds up to one

$$\sum_{c=1}^{c=b} w(c) = \sum_{c=1}^{c=b} p(c) \prod_{j=c+1}^{b} (1 - p(j)) = 1 \; .$$

As p_ν increases from zero to one, the weight factors are shifted from $c = 1$ to $c = b$, hence, the sum

$$\sum_{c=1}^{c=b} \frac{n\, w(c)}{c}$$

decreases as p_s increases.

Lemma 3. The expected number of additional routes due to time window constraints $E[m_{\nu\mu}(n)]$ is bounded between $(0, n - n/b)$. The value of $E[m_{\nu\mu}(n)]$ is a fraction of the number of customers.

Proof. By substitution, it can be shown that $E[m_{\nu\mu}(n)] = n - n/b$ when $p_\nu = 0$ and $E[m_{\nu\mu}(n)] = 0$ when $p_\nu = 1$. Since $E[m_{\nu\mu}(n)]$ is a decreasing function it is bounded between $(0, n - n/b)$.

Theorem 1. A routing problem with customers drawn from a probability measure ν has a unique p_ν such that $E[m_{\nu\mu}(n)] = n(\gamma_\nu - \gamma_\mu)$ as $n \to \infty$.

Proof. Asymptotically, the additional number of routes is $m_{\nu\mu}(n) = n(\gamma_\nu - \gamma_\mu)$ with $0 \le (\gamma_\nu - \gamma_\mu) \le 1$. Due to Lemma 2 and 3, $E[m_{\nu\mu}(n)]$ is a continuously decreasing function. Hence, there is a unique p_ν such that $E[m_{\nu\mu}(n)] = n(\gamma_\nu - \gamma_\mu)$.

Corollary. The value of $1 - p_\nu$, the average probability of "failing" to sequence a customer with time window constraints provides a measure, in a scale $(0,1)$, of the impact of time window constrains on VRP distance. As $1 - p_\nu$ increases the relative impact of time windows constraints on the number of routes and the distance traveled increases.

4. APPROXIMATING VRP DISTANCES

This section provides an approximation to VRP distance assuming a distribution center that serves a set of $N = \{1, 2, ..., n\}$ customers on any given day or time period. The number of daily requests may vary but it never exceeds $\overline{n}$, i.e. $n \leq \overline{n}$. The total number of customers with time windows is denoted n_t, $n_t \leq n$, and the total demand is denoted $q_N = \sum_{i \in N} q_i$. The focus of this research is the derivation of general approximations to the average distance traveled to serve a total of n customers with n_t time windows, $1 \leq n \leq \overline{n}$ and $0 \leq n_t \leq n$. This average distance is denoted $VRP(n, n_t, \nu)$. Instances of daily demands are formed by joining n_t customers, drawn according to a probability measure ν, and $n - n_t$ customers drawn according to probability measure μ. A customer has a time window if either $e_i > e_0$ or $l_i < l_0$.

The value of p_ν is approximated as the value that minimizes the absolute value of the difference:

$$\min | m_\nu(\overline{n}) - \sum_{c=1}^{b} \frac{\overline{n}\, p(c)}{c} \prod_{j=c+1}^{b} (1 - p(j)) | \qquad (2)$$

s.t.: $p(c) = (p_\nu)^{c-1}$, $0 \leq p_\nu \leq 1$, and $b \approx \tilde{b} = \overline{n} / m_\mu(\overline{n})$.

From Theorem 1, it is guaranteed that there is only one p_ν that minimizes the absolute value of (2). The value of $m_\mu(\overline{n})$ and $m_\nu(\overline{n})$ can be estimated by sampling from the respective distributions and determining the number of routes need.

To estimate the number of additional routes due to time windows when $0 < n_t < n$, it is necessary to model how time windows are distributed among routes. Assuming a binomial distribution, the probability of having a route with k time windows out of c customers is:

$$binomial(k; c, p_{n_t}) = \binom{c}{k} p_t^{\,k} (1 - p_t)^{c-k}$$

where:

$$\binom{c}{k} = \frac{c!}{k!(c-k)!},$$

$k =$ number of successes in b trials,

$c =$ number of independent trials, and

$p_t = n_t / n =$ the probability of success on each trial.

Then, the number of additional routes to serve a total of n customers with n_t time windows can be approximated as follows:

$$m_{\nu\mu}(n,n_t) = \sum_{c=1}^{b} \frac{n\, p(c,n,n_t,\nu)}{c} \prod_{j=c+1}^{b} (1 - p(j))\ ] - m_\mu(n)$$

where $p(c,n,n_t,\nu) = \sum_{k=0}^{c} binomial(k; c, p_{n_t})\,(p_s)^{k-1}$.

Approximating the number of routes related to "bin-packing" constraints such as vehicle capacity or tour duration is relatively straightforward:

$$m_\mu(n) \approx \max(\left\lceil \frac{q_N}{Q} \right\rceil, \left\lceil \frac{n\,\tau}{l_0 - e_0} \right\rceil)$$

where τ is the sum of estimated travel time plus service time per customer.

Although asymptotic results indicate that number of routes is the only essential factor to estimate VRP distances, the literature review has shown that the best approximations to VRP distance account for (a) a term related to the distance traveled between the depot and customers and (b) a term related to the distance traveled between customers. The proposed approximation (3) also accounts for both types of distances but adding terms to estimate the additional impact of time windows.

$$VRP(n,n_t,\nu) \approx \tilde{k}_\mu \sqrt{nA} + \tilde{k}_{\lambda\mu} \sqrt{n_t A} + \tilde{k}^m_\mu \tilde{\gamma}_\mu 2r\, m_\mu(n) + \tilde{k}^m_{\nu\mu} 2r\, m_{\nu\mu}(n,n_t) \qquad (3)$$

The vector of coefficients $(\tilde{k}_\mu, \tilde{k}_{\lambda\mu}, \tilde{k}^m_\mu, \tilde{k}^m_{\nu\mu})$ is estimated by linear regression. The coefficients $\tilde{k}^m_\mu$ and $\tilde{k}^m_{\nu\mu}$ are related to the distance generated by the number of routes needed; the coefficients $\tilde{k}_\mu$ and $\tilde{k}_{\lambda\mu}$ are related to the intercustomer distance, as in Beardwood et al. (1959). If $\tilde{k}_\nu \sqrt{n_t A}$ and $\tilde{k}_\mu \sqrt{nA}$ approximate the intercustomer distance with and without time windows respectively, then, $\tilde{k}_{\nu\mu}$ represents the *change* in intercustomer distance when time window constraints are added:

$\tilde{k}_\nu \sqrt{n_t A} = \tilde{k}_\mu \sqrt{nA} + \tilde{k}_{\nu\mu} \sqrt{n_t A}$ where $\tilde{k}_{\nu\mu} = \tilde{k}_\nu - \tilde{k}_\mu$ and $n_t = n$.

The other two remaining coefficients relate to the number of routes as follows:

$\tilde{\gamma}_\mu m_\mu(n) \to n\gamma_\mu$ as $n \to \infty$, and

$\tilde{\gamma}_{\nu\mu} m_{\nu\mu}(n,n_t) \to n(\gamma_\nu - \gamma_\mu)$ as $n \to \infty$.

The next section describes the experimental setting where approximation (3) is tested.

5. EXPERIMENTAL SETTING AND RESULTS

The experimental setting is based on the classical instances of the VRP with time windows proposed by Solomon (1987). The Solomon instances include distinct spatial customer distributions, vehicles' capacities, customer demands, and customer time windows. These problems have not only been widely studied in the operations research literature but the datasets are readily available.

The well-known 56 Solomon benchmark problems for vehicle routing problems with hard time windows are based on six groups of problem instances with 100 customers. The six problem classes are named C1, C2, R1, R2, RC1, and RC2. Customer locations were randomly generated (problem sets R1 and R2), clustered (problem sets C1 and C2), or mixed with randomly generated and clustered customer locations (problem sets RC1 and RC2). Problem sets R1, C1, and RC1 have a shorter scheduling horizon, tighter time windows, and fewer customers per route than problem sets R2, C2, and RC2 respectively. Random samples of the Solomon problems are used to examine the accuracy of models. Out of $N = 100$ possible customers in a service area A, a problem or instance is formed by a subset of n randomly selected customers. Using the first instance of the six problem types proposed by Solomon, 15 subsets of customers of sizes 70, 60, 50, 40, 30, 20, and 10 were randomly selected from the original 100 customers; $\bar{n} = 70$. All problem instances in this research were solved with a VRP improvements heuristic that has obtained the best published solution in terms of number of vehicles (Figliozzi, 2008a).

Real-world routes have a relatively small number of customers per route due to capacity, time windows, or tour length constraints (Figliozzi et al., 2007). For example, in Denver over 50% of single and combination truck routes include less than 6 stops (Holguin-Veras and Patil, 2005) and 95% of the truck routes include less than 20 stops. This research tests the proposed VRP distance approximation in instances that range from 1 to over 35 customers per route. To obtain this range of customers per route, new instances were systematically created varying the levels of customer demand and the percentages of customers with time windows.

To test different levels of customer demand, new instances were created applying the demand factors presented in Table 1 to each subset of customers. Applying the factors in the second row of demand factors in Table 1, the customers have similar demands as in the original Solomon problems (the row characterized by all ones [1]). The resulting problems using the highest demand multipliers (last row of Table 1) are such that some customers are truckload (TL) or almost TL customers. Increasing some customer demands to or close to the

TL level was done in order to test the approximation when problems are highly constrained and have a large number of routes. In addition, for each sample, out of the *n* customers a random subset of time windows is turned off; the percentage of customers with time windows ranges from 0 to 100%, in increments of 20%. In all cases, route durations were limited by the depot time window. For each Solomon problem class, variability is introduced in three distinct ways: a) different subsets of customer locations, b) different levels of customer demands, and c) different percentages of time window constraints.

Analysis of Experimental Results

All the regression results presented in this section are obtained forcing the intercept or constant term to be zero; this is consistent with previous studies by Chien (1992) and Kwon et al. (1995).

The average probabilities $(1-p_v)$ of failing to connect any two customers due to time window constraints are shown in Table 2. The values of $(1-p_v)$ do reflect the characteristics of the underlying problem types. Type 1 problems where time windows are tight result in higher $(1-p_v)$ values. Table 2 also provides an understanding of the relative impact of time window constraints on distance traveled. As the level of demand increases, the relative size of the "bin" or vehicle capacity is reduced and there is a consequent reduction in the feasible number of customers per route. Hence, the impact of time window constraints is reduced as capacity constraints become more "binding".

The estimated regression parameters disaggregated by problem type are shown in Table 3. These parameters are obtained by pooling the data of all different demand levels per problem type, i.e. using one set of parameters $(\tilde{k}_\mu, \tilde{k}_{\lambda\mu}, \tilde{k}_\mu^m, \tilde{k}_{v\mu}^m)$ for all instances. It is reassuring that the regression parameters are not only statistically significant but also reflect the characteristics of the underlying problem types. The values of $\tilde{k}_\mu$ are lowest and highest for clustered and random problems respectively. In all cases the coefficients $\tilde{k}_{\lambda\mu}$ are significant and positive which suggests that time window constraints increase the distance traveled between customers. The coefficients $\tilde{k}_{\lambda\mu}$ follow a similar trend as the $\tilde{k}_\mu$ coefficients; lowest and highest values for clustered and random problems respectively. As expected, the values of $\tilde{k}_\mu^m$ are slightly less than one but significantly different than zero. The type C2 coefficients demonstrate that although $\tilde{k}_{v\mu}^m$ is zero, $\tilde{k}_{\lambda\mu}$ can be positive and significant; i.e. time window constraints increase the intercustomer distance but do not affect the number of routes that is determined by capacity constraints.

To evaluate the prediction accuracy, the MPE (Mean Percentage Error) and the $MAPE$ (Mean Absolute Percentage Error) are used and calculated as follows:

$$MPE = \frac{1}{p}\sum_{i=1}^{p}\frac{D_i - E_i}{D_i} * 100\%$$

$$MAPE = \frac{1}{p}\sum_{i=1}^{p}\frac{|D_i - E_i|}{D_i} * 100\%$$

Where the actual distance for instance i is denoted D_i and the estimated distance is denoted E_i. For a given set of instances it is always the case that $MPE \le MAPE$. The MPE indicates whether the estimation, on average, overestimates or underestimates the actual distance. The $MAPE$ provides the average deviation between actual and estimated distance as a percentage of the actual distance.

Model fit R^2, *MAPE*, and *MPE* are displayed for each problem class and pooled data in Table 4**.** The approximation quality is high, particularly for random and random clustered problems. The values of *MAPE* range from 3.4 to 5.6% with an average of 4.5% for the pooled data. As expected, a better fit can be obtained if a regression is run for each demand level. Approximation quality, as evaluated by *MAPE*, improves significantly as shown in Table 5.

6. CONCLUSIONS

This is the first research effort to study and test approximations to the average length of vehicle routing problems when there is variability in the number of customers, time window constraints, and demand levels. Based on asymptotic properties of vehicle routing problems, a probabilistic modeling approach was developed to approximate the average distance traveled. An expression to estimate the number of additional routes needed for a varying number of time windows constraints is derived using the average probability of successfully sequencing a customer with time windows. It is proven that this average probability is a unique characteristic of a vehicle routing problem. This probability also indicates the relative importance of time windows constraints on VRP distances. The experimental results demonstrate that the quality of the approximation is robust in terms of *MAP* and *MAPE*. In addition, the estimated regression parameters are intuitive and reflect the characteristics of the underlying routing problems.

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TABLES

Table 1. Demand Factors

Problem		C1	R1	CR1	C2	R2	RC2
Vehicle Capacity		200	200	200	700	1000	1000
Max. Demand		50	41	40	41	41	40
Demand Level	0	0.00	0.00	0.00	0.00	0.00	0.00
	1	1.00	1.00	1.00	1.00	1.00	1.00
	2	1.60	1.78	1.80	3.60	5.68	5.80
	3	2.20	2.56	2.60	6.20	10.36	10.60
	4	2.80	3.34	3.40	8.80	15.04	15.40
	5	3.40	4.12	4.20	11.40	19.72	20.20
	6	4.00	4.90	5.00	14.00	24.40	25.00

Table 2. Average Probability $(1-p_s(\nu))$

		C1	R1	RC1	C2	R2	RC2
Demand Level	0	13.6%	25.4%	20.5%	0.0%	9.1%	10.6%
	1	13.6%	25.4%	20.5%	0.0%	9.1%	10.6%
	2	1.2%	23.1%	7.0%	0.0%	1.0%	1.0%
	3	0.0%	9.8%	1.4%	0.0%	0.0%	0.0%
	4	0.0%	2.6%	0.6%	0.0%	0.0%	0.0%
	5	0.0%	1.8%	0.0%	0.0%	0.0%	0.0%
	6	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%

Table 3. Estimated Regression Coefficients by Problem Class

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Problem	$\tilde{k}_{\mu}$	t-stat	$\tilde{k}_{\lambda\mu}$	t-stat	$\tilde{k}_{\mu}^{m}$	t-stat	$\tilde{k}_{\nu\mu}^{m}$	t-stat
C101	0.67	24.3	0.25	5.6	0.81	48.5	1.31	6.49
R101	0.89	54.6	0.32	8.0	0.80	78.5	1.02	24.42
RC101	0.70	40.8	0.21	4.1	0.94	108.2	0.98	16.51
C201	0.70	37.5	0.29	10.1	0.78	55.7	-----	-----
R201	0.99	48.1	0.41	17.1	0.73	57.9	1.44	16.84
RC201	0.77	48.8	0.34	12.5	0.88	106.9	1.54	25.87

Table 4. Approximation Quality by Problem Class (Pooled data)

Problem	R2	MAPE	MAP
C101	0.996	5.6%	-0.6%
R101	0.999	3.4%	-0.3%
RC101	0.999	3.3%	-0.5%
C201	0.995	5.8%	-1.8%
R201	0.997	5.2%	-0.2%
RC201	0.999	4.0%	-0.7%

Table 5. Average Approximation Quality by Problem Class (By Distribution)

Problem	R2	MAPE	MAP
C101	0.998	3.9%	-0.7%
R101	0.999	2.5%	0.4%
RC101	1.000	1.9%	-0.4%
C201	0.994	5.5%	0.3%
R201	0.999	3.1%	-0.4%
RC201	0.999	2.6%	-0.3%

APPENDIX A

Lemma 2. The expected number of additional routes due to time window constraints, $E(m_{\nu\mu})$, is a continuously decreasing function of p_ν .

$$E(m_{\nu\mu}) = [\sum_{c=1}^{c=b} \frac{n\, p(c)}{c} \prod_{j=c+1}^{b} (1 - p(j))] - \frac{n}{b}$$

Proof. This is a continuous function because it is a linear combination of continuous functions of the variable p_ν.

The weight factors:

$$p(c) \prod_{j=c+1}^{b} (1 - p(j))$$

are applied to each feasible route with c customers per route. Developing the sum of weight factors and denoting $p(c) = p_c$ for the sake of brevity:

$$\sum_{c=1}^{c=b} p_c \prod_{j=c+1}^{b} (1 - p_j) =$$

$$= p_b + p_{b-1}(1 - p_b) + ... + p_2 (1 - p_b)(1 - p_{b-1})...(1 - p_3) + p_1 (1 - p_b)(1 - p_{b-1})...(1 - p_2) =$$

making p_b a common factor:

$$= p_b[1 + p_{b-1}(\frac{1 - p_b}{p_b} + ... + p_2 \frac{1 - p_b}{p_b}(1 - p_{b-1})...(1 - p_3) + p_1 \frac{1 - p_b}{p_b}(1 - p_{b-1})...(1 - p_2)] =$$

making $\frac{1 - p_b}{p_b}$ a common factor:

$$= p_b[1 + \frac{1 - p_b}{p_b}[p_{b-1} + ... + p_2 (1 - p_{b-1})...(1 - p_3) + p_1 (1 - p_{b-1})...(1 - p_2)]] =$$

The term $p_b[1 + \frac{1 - p_b}{p_b}] = 1$, then:

$$\sum_{c=1}^{c=b} p_c \prod_{j=c+1}^{b} (1 - p_j) = p_{b-1} + ... + p_2 (1 - p_{b-1})...(1 - p_3) + p_1 (1 - p_{b-1})...(1 - p_2)$$

The same process can be continued until:

$$\sum_{c=1}^{c=b} p_c \prod_{j=c+1}^{b} (1 - p_j) = p_2 + p_1 (1 - p_2) = 1 .$$

Since $p_1 = 1$ as all customers can be served from the depot without violating time window constraints, this proves that:

$$\sum_{c=1}^{c=b} p_c \prod_{j=c+1}^{b} (1 - p_j) = 1$$

Developing the sum and replacing $p_c = (p_v)^{c-1} = p^{c-1}$ for the sake of brevity, for any sum up to $b = i$:

$$\sum_{c=1}^{c=i} \frac{n\,p(c)}{c} \prod_{j=c+1}^{i} (1 - p(j)) =$$

$$n[\frac{p^{i-1}}{i} + \frac{p^{i-2}}{i-1}(1-p^{i-1}) + \frac{p^{i-3}}{i-2}(1-p^{i-1})(1-p^{i-2}) + \ldots$$

$$\ldots + \frac{p^1}{2}(1-p^{i-1})(1-p^{i-2})\ldots(1-p^2) + \frac{p^0}{1}(1-p^{i-1})(1-p^{i-2})\ldots(1-p^2)(1-p^1)] =$$

Making the terms $1 - p^j,\ j = 1,\ldots,i-1$ common factors:

$$n[\frac{p^{i-1}}{i} + [(1-p^{i-1})[\frac{p^{i-2}}{i-1} + [(1-p^{i-2})[\ldots.\ \ \ldots + [\frac{p^2}{3} + [(1-p^2)[\frac{p^1}{2} + [(1-p^1)\frac{p_1}{1}]]\ldots]]$$

For any increase in p_v, any $p^j,\ j = 1,\ldots,i-1$ will have an increase. However, any increase in p^j will reduce the term $1 - p^j$ that multiplies the sum of the $j-1, j-2, \ldots., 2, 1$ terms. Since the sum of the weight factors remains constant and equal to one, as p_v increases the weight applied to the terms:

$$n\frac{p^{j-2}}{j-1}, n\frac{p^{j-3}}{j-2}, \ldots., n\frac{p^1}{2}, n1$$

decreases whereas the term $n\dfrac{p^{j-1}}{j}$ increases for any $j = 1,\ldots,i-1$. Hence, as p_v increases $E(m_{v\mu})$ decreases. In particular, as p_v increases the term with the largest index always increases whereas the term with index one always decreases.