EE 520: Random Processes **Fall 2020**

Lecture 15
Counting Processes

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Recommended Reading: Pishro-Nik: 11.1.1 - 11.2.5; Gubner: 11.1 - 11.4

We will spend the remainder of the quarter studying the most common random processes (RPs) and their properties. One general class of RPs is that of *counting processes*.

Definition 1. A **counting process** $\{N_t, t \geq 0\}$ is a RP that counts how many times an event occurs from time zero up to time t.

Example 1. N_t could denote the number of hits to a website up to and including time t.

Definition 2. Let $0 \leq t_1 \leq t_2 < \infty$ be times. Then $N_{t_2} - N_{t_1}$ denotes the number of occurrences from time t_1 to t_2. We call $N_{t_2} - N_{t_1}$ an **increment** of the RP $\{N_t\}$.

1 Poisson Processes

The counting process we'll study most closely in this course is the *Poisson process*, defined below. A Poisson RP is one where the *increments* are Poisson, but it turns out you can use this fact to show that N_t itself is also a Poisson RV.

Definition 3. A counting process $\{N_t, t \geq 0\}$ is called **Poisson** if

1. $N_0 = 0$
2. For any $0 \leq s < t < \infty$, the increment $N_t - N_s$ is a Poisson RV with parameter $\lambda(t-s)$, i.e.,
$$P\left(N_t - N_s = k\right) = \frac{\left(\lambda(t-s)\right)^k e^{-\lambda(t-s)}}{k!},\ k = 0, 1, 2, \ldots$$
3. If the time intervals $(t_1, t_2], (t_2, t_3], \ldots, (t_n, t_{n+1}]$ are disjoint, then the increments $N_{t_2} - N_{t_1}, N_{t_3} - N_{t_2}, \ldots, N_{t_{n+1}} - N_{t_n}$ are independent. In this case, we call them *independent increments*.

Definition 4. The constant λ in the PDF of the increments is called the **rate** or **intensity** of the process.

Intuitively, the rate governs how many arrivals occur in a fixed period of time, so a higher rate corresponds to more arrivals. Next we consider the time at which a certain number of arrivals have occurred.

Definition 5. The **jump times** or **arrival times** are the times at which n events have occurred

$$T_n = \min\left\{t > 0 : N_t \geq n\right\}.$$

Fact 1. The nth jump time has distribution

$$f_{T_n}(t) = \lambda \frac{(\lambda t)^{n-1} e^{-\lambda t}}{(n-1)!},\ t \geq 0.$$

Proof. Note that

$$P\left(T_n > t\right) = P\left(N_t < n\right) = \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!} e^{-\lambda t}.$$

Now note that $F_{T_n}(t) = 1 - P(T_n > t)$ and differentiate to obtain the given PDF. □

Definition 6. The **interarrival times** are the lengths of time between each occurrence

$$X_n = \begin{cases} T_1, & n = 1 \\ T_n - T_{n-1}, & n = 2, 3, \ldots \end{cases}$$

Fact 2. The interarrival times are i.i.d. and have an exponential distribution

$$X_n \overset{\text{i.i.d.}}{\sim} \exp(\lambda), \quad f_{X_n}(x) = \lambda e^{-\lambda x},\ x \geq 0.$$

We will not prove this fact, but you can use the definition and the distribution of the jump times to see that it holds for $n = 1$. Note the relationship between the jump/arrival times and interarrival times gives

$$T_n = X_1 + \cdots + X_n.$$

Keeping the number of arrivals, increments, jump times, and interarrival times organized can be difficult, so take care when working problems and/or creating your equation sheet for the final exam.

1.1 Moments of Poisson Processes

Let $\{N_t\}$ be a Poisson RP. then

$$\mathbb{E}[N_t] = \mathbb{E}[N_t - N_0] = \lambda t,$$

where we have used the fact that $N_t \sim \text{Poisson}(\lambda t)$ and $N_0 = 0$. Since N_t is Poisson (why?), we can easily obtain

$$\begin{aligned} \text{var}(N_t) &= \lambda t \\ \mathbb{E}\left[N_t^2\right] &= \lambda t + (\lambda t)^2. \end{aligned}$$

Next we examine the correlation function. Let $0 \leq s < t$. Then

$$\begin{aligned} \mathbb{E}\left[N_t N_s\right] &= \mathbb{E}\left[(N_t - N_s)N_s\right] + \mathbb{E}\left[N_s^2\right] \\ &= \mathbb{E}\left[(N_t - N_s)\left(N_s - N_0\right)\right] + \mathbb{E}\left[N_s^2\right] \\ &= \mathbb{E}\left[N_t - N_s\right]\mathbb{E}\left[N_s - N_0\right] + \mathbb{E}\left[N_s^2\right] \\ &= \lambda(t - s)\lambda s + (\lambda s)^2 + \lambda s \\ &= \lambda s(1 + \lambda t). \end{aligned}$$

We can use this to also see that

$$\text{cov}\left(N_t, N_s\right) = \mathbb{E}\left[N_t N_s\right] - \mathbb{E}\left[N_t\right]\mathbb{E}\left[N_s\right] = \lambda s.$$

More generally, we have

$$\text{cov}\left(N_t, N_s\right) = \lambda \min(t, s).$$

2 Wiener Processes

The *Wiener process* or *Brownian motion* is a RP that models integrated white noise, which would otherwise be very difficult to deal with.

Definition 7. A counting process $\{W_t, t \geq 0\}$ is called a **Wiener process** if

1. $W_0 = 0$
2. For any $0 \leq s < t < \infty$, the increment $W_t - W_s$ is a Gaussian RV with zero mean and variance $\sigma^2(t - s)$.

3. If the time intervals $(t_1, t_2], (t_2, t_3], \dots, (t_n, t_{n+1}]$ are disjoint, then the increments $W_{t_2} - W_{t_1}, W_{t_3} - W_{t_2}, \dots, W_{t_{n+1}} - W_{t_n}$ are independent. In this case, we call them *independent increments*.

4. For each point in the sample space $\omega \in \Omega$, $W_t(\omega)$ is a continuous function of t. As a result, we say that W_t has *continuous sample paths*.

Fact 3. For a Wiener process, $\text{cov}(W_t, W_s) = \sigma^2 \min(t, s)$.

Based on the above, is a Wiener process WSS?

3 Gaussian Processes

Definition 8. A continuous-time RP $\{X_t\}$ is called **Gaussian** if for every sequence of times $t_1, \dots, t_n$ the random vector $[X_{t_1} \ \dots \ X_{t_n}]^T$ is Gaussian, or equivalently if every finite linear combination of samples $\sum_{i=1}^n c_i X_{t_i}$ is a Gaussian RV.

Fact 4. A Wiener RP is a Gaussian RP.

Proof. For any $0 < t_1 < \cdots < t_n$, write

$$\begin{bmatrix} W_{t_1} \\ W_{t_2} \\ \vdots \\ W_{t_n} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 0 \\ 1 & 1 & 1 & \dots & 0 \\ \vdots & & \ddots & \vdots & \\ 1 & 1 & 1 & \dots & 1 \end{bmatrix} \begin{bmatrix} W_{t_1} - W_0 \\ W_{t_2} - W_{t_1} \\ \vdots \\ W_{t_n} - W_{t_{n-1}} \end{bmatrix}.$$

Since $W_{t_i} - W_{t_{i-1}} \sim \mathcal{N}(0, \sigma^2)$, the above is a linear combination of i.i.d. Gaussians and therefore a Gaussian RVec. ☐