EE 520: Random Processes **Fall 2021**

Lecture 13
Random Processes

Instructor Name: John Lipor

Recommended Reading: Pishro-Nik: 10.1.0 - 10.1.4, 10.2.0 - 10.2.2; Gubner: 10.1 - 10.4

1 Random Processes

To this point, we have considered only finite collections of RVs, either as RVs directly or as random vectors. The remainder of the course will focus on the case where we have an infinite collection of RVs, which we call a *random process* (RP).

Definition 1. A **discrete time random process** is a countable collection of RVs

$$\{X_n \in \mathbb{R} : n \in S\},$$

where S is a countably infinite set. Typically, we take $S = \mathbb{N}$ (the natural numbers).

Example 1. Assume a sequence of bits is transmitted over a noisy channel and bits are flipped independently with probability p, so that $\{X_n : n \in \mathbb{N}\}$ is a Ber(p) random process.

Definition 2. A **continuous time random process** is an uncountable collection of RVs

$$\{X_t \in \mathbb{R} : t \in \mathcal{T}\},$$

where $\mathcal{T}$ is an uncountable subset of $\mathbb{R}$. Typically we take $\mathcal{T} = [0, \tau]$ or $\mathcal{T} = \mathbb{R}$.

Example 2. Let $\{N_t : t \in \mathcal{T}\}$ count the number of occurrences of some event of interest up to time t. This is known as a *counting process* and is a continuous time RP.

1.1 Relation to Sample Space

Recall that a RV is a function from the sample space to $\mathbb{R}$. For a RP, we really have a set $\{X_n(\omega)\}$, and we can view a RP in two ways:

1. Fix n, then view each $X_n(\omega)$ as a RV.
2. Fix ω, which gives the sequence $X_1(\omega), X_2(\omega), \ldots$. This sequence is called a *realization* or *sample path* of the RP.

The second view above is more common and aligns better with the notion of a realization of a RV.

Example 3. Let $(\Omega, \mathcal{F}, P) = ([0, 2\pi], \mathcal{B}([0, 2\pi]), \text{Unif}([0, 2\pi]))$, where $\mathcal{B}(A)$ denotes the Borel sigma algebra generated by the set A. Consider the RP

$$X_t(\omega) = \cos(2\pi f_0 t + \omega).$$

Under View 1 above, we think of the RP in terms of the possible RVs $X_t(\omega)$ for a fixed time t. Under View 2, we think of possible sample paths that would result from different values of ω, as depicted in Fig. 1 below.

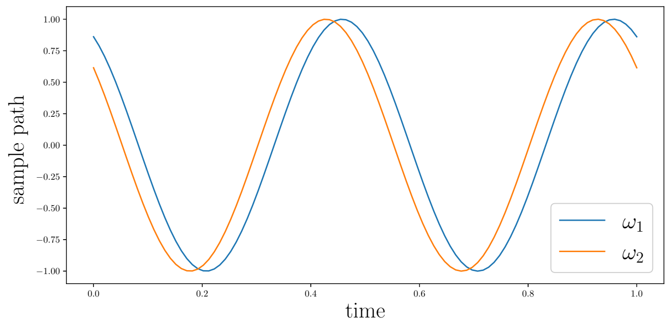

Figure 1: Examples of two sample paths for different values of ω.

2 Characterization of RPs

For RVs and RVecs, we spent a great deal of time explicitly writing down distributions, from which we can derive parameters such as the mean, variance, and covariance. We will discuss the analog for RPs next week. For now, we will focus on the first (mean) and second-order (variance, covariance) statistics of RPs.

Definition 3. For a RP $\{X_t\}$ (or X_n), the **mean function** is

$$m_X(t) = \mathbb{E}[X_t].$$

Definition 4. For a RP X_t, the **correlation function** or **autocorrelation** between two RVs X_t and X_s is

$$R_X(t,s) = \mathbb{E}[X_t X_s].$$

Example 4. Let $X_t = \cos(2\pi f t + \theta)$ where $\theta \sim \text{Unif}([-\pi,\pi])$. Then

$$\begin{aligned}\mathbb{E}[X_t] &= \mathbb{E}\left[\cos(2\pi f t + \theta)\right] \\ &= \int_{-\pi}^{\pi} \frac{1}{2\pi}\cos(2\pi f t + \theta)\, d\theta \\ &= 0,\end{aligned}$$

where the last line follows since a cosine integrated over an entire period is zero. For correlation, we get

$$\begin{aligned}R_X(t,s) &= \mathbb{E}\left[X_t X_s\right] \\ &= \mathbb{E}\left[\cos(2\pi f t + \theta)\cos(2\pi f s + \theta)\right] \\ &= \frac{1}{2}\mathbb{E}\left[\cos(2\pi f(t+s) + 2\theta) + \cos(2\pi f(t-s))\right] \\ &= \frac{1}{2}\cos(2\pi f(t-s)).\end{aligned}$$

Some important properties of the correlation function are as follows.

1. Symmetry: $R_X(t,s) = R_X(s,t)$
2. Positive semidefiniteness: For any function $\alpha(t)$,

$$\int\int \alpha(t) R_X(t,s)\alpha(s)dtds \geq 0.$$

This generalizes the idea that correlation/covariance matrices are positive semidefinite.

3. The Cauchy-Schwarz inequality for RPs states that

$$|R_X(t,s)| = |\mathbb{E}[X_t X_s]| \le \sqrt{\mathbb{E}[X_t^2 X_s^2]} = \sqrt{R_X(t,t) R_X(s,s)}.$$

Some other second-order quantities of interest are given below.

Definition 5. A RP with $\mathbb{E}\left[X_t^2\right] = R_X(t,t) < \infty$ is called a **second-order process**.

Definition 6. For a RP $\{X_t\}$, the **covariance function** between X_t and X_s is

$$C_X(t,s) = \mathbb{E}\left[(X_t - \mathbb{E}[X_t])(X_s - \mathbb{E}[X_s])\right].$$

Definition 7. For two RPs $\{X_t\}$ and $\{Y_t\}$, the **cross-correlation function** is

$$R_{XY}(t,s) = \mathbb{E}[X_t Y_s].$$

Definition 8. For two RPs $\{X_t\}$ and $\{Y_t\}$, the **cross-covariance function** is

$$\begin{aligned} C_{XY}(t,s) &= \mathbb{E}\left[(X_t - \mathbb{E}[X_t])(Y_s - \mathbb{E}[Y_s])\right] \\ &= R_{XY}(t,s) - m_X(t) m_Y(s). \end{aligned}$$

3 Stationarity

Stationarity captures the notion that the statistics of a RP do not change over time. We typically consider two definitions of stationarity (strict and wide), and the latter appears more frequently in practice.

Definition 9. A RP is n**th-order strictly stationary** if for any finite collection of n times $t_1, \ldots, t_n$, all joint probabilities do not depend on the time shift δt, i.e.,

$$P\left(X_{t_1}, X_{t_2}, \ldots, X_{t_n} \in B\right) = P\left(X_{t_1+\delta t}, X_{t_2+\delta t}, \ldots, X_{t_n+\delta t} \in B\right).$$

Definition 10. A RP is called **strictly stationary** if it is nth-order strictly stationary for all finite $n \in \mathbb{N}$.

Example 5. Let $X_t = Z$ for all time t. Then $\{X_t\}$ is a strictly stationary RP.

In general, strict stationarity is too strong of an assumption to use, and it is very difficult to prove that it holds for interesting RPs. We instead focus mainly on wide-sense stationary (WSS) RPs.

Definition 11. A RP $\{X_t\}$ is **wide-sense stationary** (WSS) if

(i) $\mathbb{E}[X_t] = \mathbb{E}[X_s]$ for all s, t (i.e., the mean does not change over time)

(ii) $\mathbb{E}[X_t X_s]$ depends on t, s only through their difference $t - s$.

For WSS RPs, we often write $R_X(t,s)$ as a univariate function of the difference $\tau = t - s$, i.e., we write $R_X(\tau)$.

Example 6. Our previous example $X_t = \cos(2\pi f t + \theta)$ is WSS, since we showed that

$$R_X(t,s) = \frac{1}{2}\cos\left(2\pi f(t-s)\right),$$

which depends on t and s only through their difference.

4 WSS RPs Through LTI Systems

In signal processing, we often consider what happens when we pass a WSS process (which we think of as a signal) through a linear time-invariant (LTI) system (e.g., a filter). As with the deterministic signals we study in a typical course on discrete-time signal processing, WSS signals are often more convenient to analyze in the Fourier domain.

Definition 12. The **power spectral density** (PSD) of a WSS process is the Fourier transform of the correlation function

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f\tau} d\tau.$$

We can also invert the PSD to find the autocorrelation

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f) e^{j2\pi f\tau} df.$$

Theorem 1. Any real, symmetric power spectral density $R_X(\tau)$ satisfies

1. $R_X(0) \geq |R_X(\tau)|$
2. $S_X(f)$ is real and symmetric
3. $S_X(f) \geq 0$

Parts of the proof may appear on the homework. Now consider a LTI system defined by the impulse response $h(t)$. Analyzing in the time domain, we have

$$Y_t = \int_{-\infty}^{\infty} h(\tau) X_{t-\tau} d\tau.$$

The resulting mean function is

$$\begin{aligned} m_Y(t) &= \mathbb{E}[Y_t] \\ &= \int_{-\infty}^{\infty} h(\tau)\mathbb{E}[X_{t-\tau}] d\tau \\ &= m_X(t) \int_{-\infty}^{\infty} h(\tau) d\tau, \end{aligned}$$

where $m_X(t) = m_X$ does not depend on t, since $\{X_t\}$ is WSS. The correlation function is

$$\begin{aligned} R_Y(t,s) &= \mathbb{E}[Y_t Y_s] \\ &= \mathbb{E}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau) X_{t-\tau} h(\theta) X_{s-\theta} d\theta d\tau\right] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau) h(\theta) \mathbb{E}\left[X_{t-\tau} X_{s-\theta}\right] d\theta d\tau \\ &= \int_{-\infty}^{\infty} h(\tau) \int_{-\infty}^{\infty} h(\theta) R_X\left((t-s) - (\tau-\theta)\right) d\theta d\tau, \end{aligned} \tag{1}$$

which depends on t, s only through their difference. Hence $\{Y_t\}$ is a WSS RP.

Definition 13. Let $\{X_t\}$ and $\{Y_t\}$ be two WSS RPs. If the cross-correlation $R_{XY}(t,s)$ depends on t, s only through their difference, we say $\{X_t\}$ and $\{Y_t\}$ are **jointly WSS**.

Consider passing the RP X_t through the LTI system defined by $h(t)$, yielding the output process

$$Y_t = \int_{-\infty}^{\infty} h(\theta) X_{t-\theta} d\theta.$$

Their cross-correlation becomes

$$\begin{aligned}
\mathbb{E}\left[X_t Y_s\right] &= \mathbb{E}\left[X_t \int_{-\infty}^{\infty} h(\theta) X_{s-\theta} d\theta\right] \\
&= \int_{-\infty}^{\infty} h(\theta) \mathbb{E}\left[X_t X_{s-\theta}\right] d\theta \\
&= \int_{-\infty}^{\infty} h(\theta) R_X(t-s+\theta) d\theta.
\end{aligned}$$

Letting $\tau = t - s$, we can write

$$R_{XY}(\tau) = \int_{-\infty}^{\infty} h(\theta) R_X(\tau+\theta) d\theta.$$

Substituting the above into Eq. (1), we see that

$$R_Y(\tau) = \int_{-\infty}^{\infty} h(\beta) R_{XY}(\tau-\beta) d\beta,$$

i.e., R_Y is the convolution of h and R_{XY}. This motivates examining these objects in the frequency domain.

Definition 14. For the LTI system with impulse response $h(t)$, the **transfer function** is

$$H(f) = \int_{-\infty}^{\infty} h(\tau) e^{-j2\pi f\tau} d\tau.$$

Definition 15. The **cross-power spectral density** (CPSD) of two jointly WSS RPs is the Fourier transform of the cross-correlation function

$$S_{XY}(f) = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-j2\pi f\tau} d\tau.$$

Given the cross-correlation function above, we can evaluate the CPSD of $\{X_t\}$ and $\{Y_t\}$, which also gives us the PSD of $\{Y_t\}$

$$S_{XY}(f) = H^*(f) S_X(f),$$

where $H^*(f)$ denotes the complex conjugate of $H(f)$. Further, we have

$$\begin{aligned}
S_Y(f) &= H(f) S_{XY}(f) \\
&= \left|H(f)\right|^2 S_X(f).
\end{aligned}$$