EE 520: Random Processes **Fall 2021**

Lecture 7
Transformations of Random Variables

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Recommended Reading: Pishro-Nik: 4.3, 4.1.3; Gubner: 5.2 - 5.5

1 CDF of Discrete Random Variables

Recall that for continuous RVs, the CDF and PDF are related by

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt \qquad f_X(x) = F_X'(x).$$

For discrete RVs, these become

$$F_X(x) = \sum_{i:x_i \leq x} p_X(x_i) \qquad p_X(x_j) = F_X(x_j) - F_X(x_{j-1}).$$

Note that in the discrete case, $f_X(x_j)$ is obtained by the finite difference approximation of the derivative.

Example 1. Consider a discrete RV X with PMF

$$p_X(x_j) = \begin{cases} p_0 & x_j = 0 \\ p_1 & x_j = 1 \\ p_2 & x_j = 2. \end{cases}$$

This yields the CDF

$$F_X(x) = \begin{cases} 0 & x < 0 \\ p_0 & 0 \leq x < 1 \\ p_0 + p_1 & 1 \leq x < 2 \\ 1 & x \geq 2. \end{cases}$$

Notice that

$$p_X(1) = F_X(1) - F_X(0) = p_0 + p_1 - p_0 = p_1.$$

2 Mixed Random Variables

RVs can also take on a combination of continuous and discrete distributions. These are called *mixed* RVs and have density

$$f_X(x) = \tilde{f}_X(x) + \sum_{i=-\infty}^{\infty} p_X(x_i)\delta(x - x_i),$$

where $\delta(\cdot)$ is the Dirac delta/impulse function. A common usage of a mixed RV is the case where we force a RV to be strictly nonnegative, as in the following example.

Example 2. Consider passing a Gaussian RV through a diode, so that anything below zero is set to zero. The resulting mixed distribution consists of

$$\tilde{f}_X(x) = \begin{cases} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \qquad p_X(x_i) = \begin{cases} \frac{1}{2} & x_i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

In other words, we keep a Gaussian distribution for anything positive and account for the negative values by placing a mass of 1/2 at $x_i = 0$.

3 Functions of Random Variables

The CDF is a useful tool for handling functions/transformations of RVs. Suppose we have a RV X and define $Y = g(X)$. Then

$$\begin{aligned} P(Y \le y) &= P\left(g(X) \le y\right) \\ &= P\left(g(X) \in (-\infty, y]\right) \\ &= P\left(X \in g^{-1}((-\infty, y])\right), \end{aligned}$$

where $g^{-1}(A)$ is the pre-image of A (i.e., the set of things that map to A under g), since g is not necessarily invertible. Note that even if g is not invertible, we can still handle some simple non-invertible functions.

Example 3. Let $Y = X^2$. Then for $y \ge 0$,

$$\begin{aligned} P(Y \le y) &= P(X^2 \le y) \\ &= P(-\sqrt{y} \le X \le \sqrt{y}) \\ &= \int_{-\sqrt{y}}^{\sqrt{y}} f_X(x)dx \\ &= F_X(\sqrt{y}) - F_X(-\sqrt{y}). \end{aligned}$$

Given the CDF, we can differentiate to find the PDF

$$f_Y(y) = \begin{cases} \frac{1}{2\sqrt{y}}\left(f_X(\sqrt{y}) + f_X(-\sqrt{y})\right) & y \ge 0 \\ 0 & y < 0. \end{cases}$$

There are two general methods for handling transformations of RVs. The above example is a case of a "nice" transformation and stems from the following theorem.

Theorem 1. Let $X \sim f_X$ be a continuous RV. Let $g(\cdot)$ be a strictly monotonic (increasing or decreasing), differentiable function. Then $Y = g(X)$ has PDF

$$f_Y(y) = \begin{cases} \left|\frac{d}{dy} g^{-1}(y)\right| f_X\left(g^{-1}(y)\right) & \text{if } y = g(x) \text{ for some } x \\ 0 & \text{otherwise.} \end{cases}$$

Example 4. Let $X \ge 0$ be a RV and set $Y = X^n$. Note that X^n is monotonic for $X \ge 0$. Then

$$g(x) = x^n \implies g^{-1}(y) = y^{1/n}$$

and

$$\frac{d}{dy} g^{-1}(y) = \frac{1}{n} y^{(n-1)/n}.$$

In the case of $n = 2$, this gives

$$f_Y(y) = \frac{1}{2\sqrt{y}} f_X\left(\sqrt{y}\right)$$

as in Example 3.

3.1 Generating arbitrary RVs from uniforms

When we restrict ourselves to invertible transformations, we can use the above idea to generate RVs with an arbitrary distribution given only RVs from a uniform distribution. Let $X \sim \text{Unif}([0,1])$ and $Y \sim f_Y$. Our goal is to find $g(\cdot)$ such that $Y = g(X)$. Note that

$$\begin{aligned} F_Y(y) &= P(Y \le y) \\ &= P(g(X) \le y) \\ &= P(X \le g^{-1}(Y)) \\ &= F_X(g^{-1}(y)) \\ &= g^{-1}(y), \end{aligned}$$

where the last line follows since $F_X(x) = x$ for the uniform distribution on $[0,1]$. Since $X = g^{-1}(Y)$, setting $Y = F_Y^{-1}(X)$ will result in the desired distribution. As a sanity check, we can examine the CDF of Y

$$\begin{aligned} F_Y(y) &= P\left(F_Y^{-1}(X) \le y\right) \\ &= P\left(X \le F_Y(y)\right) \\ &= F_X\left(F_Y(y)\right) = F_Y(y), \end{aligned}$$

where we use the fact that $F_Y(y) \in [0,1]$ by definition of the CDF.

3.2 Piecewise-monotone functions

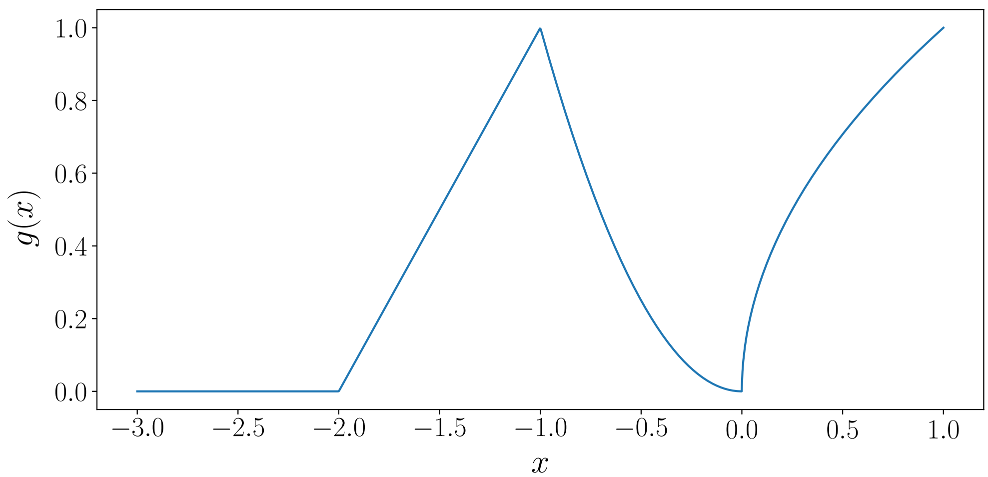
While many functions of interest will not be monotone, it may be that if we break them into small enough regions, they are monotone on each region. This approach is best illustrated through examples, but the general outline is as follows.

1. Draw a picture of $g(x)$ as a function of x.
2. Determine the "easy" regions where $F_Y(y) = 0$ and $F_Y(y) = 1$.
3. For each "interesting" region of y:
   a Draw a horizontal line at y.
   b Locate the values of x in terms of y.
   c Solve for these values of x in terms of y.
   d Use these values of x to find the set A such that $x \in A$ implies $g(x) \le y$.
   e We now have that $P(Y \le y) = P(X \in A)$.
4. Differentiate F_Y to get f_Y.

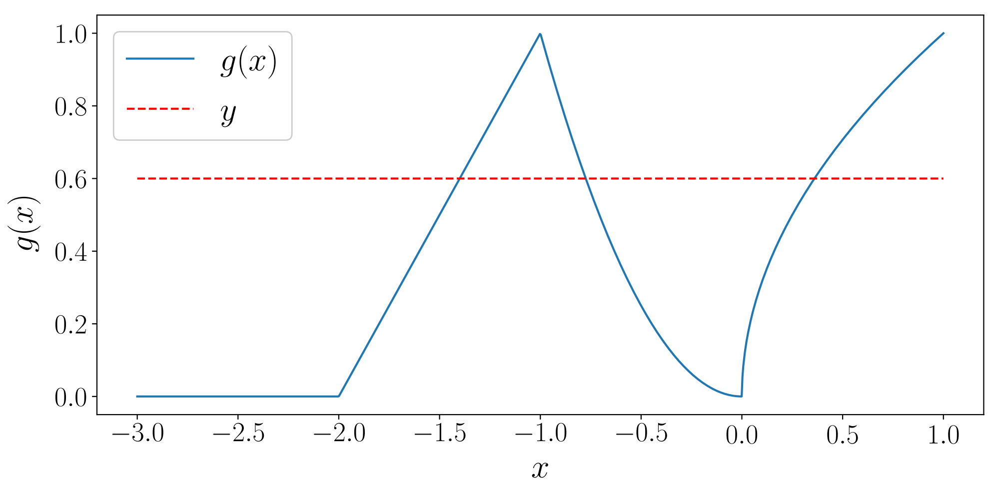
Example 5. Let $X \sim \text{Unif}([-3,1])$ and $Y = g(X)$, where

$$g(x) = \begin{cases} 0 & x < -2 \\ x+2 & -2 \le x < -1 \\ x^2 & -1 \le x < 0 \\ \sqrt{x} & x \ge 0. \end{cases}$$

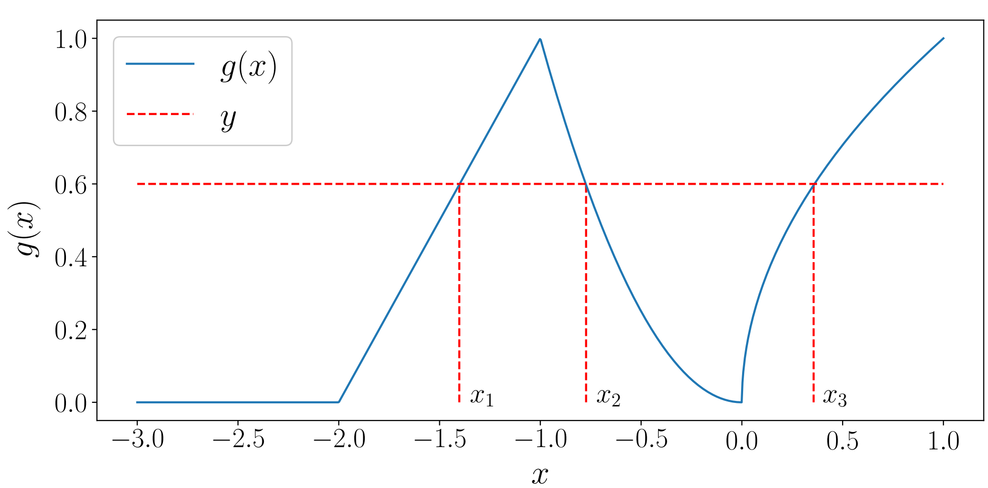
Our goal is to find the CDF of Y. Following the instructions above, we first draw a picture, shown in Fig. 1a. The first step is to take care of the "easy" parts.



(a)



(b)



(c)

Figure 1: Plots associated with Example 5.

Easy region 1: The first of these is the case of $g(x) = y < 0$, which never occurs, so we have

$$P(Y \leq y) = 0 \qquad y < 0.$$

Easy region 2: The first of these is the case of $g(x) = y \geq 1$, which is always true, so we have

$$P(Y \leq y) = 1 \qquad y \geq 1.$$

Interesting region 1: We now move on to the regions where something "interesting" happens. In this example, there is only one of these, corresponding to the case where $0 \leq y < 1$. We fix an arbitrary $y \in [0, 1)$ and find all the locations where $g(x)$ intersects the horizontal line at y, as depicted in Fig. 1b.

$$P(Y \leq y) = 1 \qquad y \geq 1.$$

Now we draw vertical lines at each location where the two lines intersect, as depicted in Fig. 1c. From the figure, we see that we have $g(x) \leq y$ for $x \leq x_1$ and $x_2 \leq x \leq x_3$.

1. In the region of x_1, we have $g(x) = x + 2$, so solving gives $x_1 = y - 2$.
2. In the region of x_2, we have $g(x) = x^2$, so solving gives $x_2 = -\sqrt{y}$, where we have used the plot to infer that $x < 0$.
3. In the region of x_3, we have $g(x) = \sqrt{x}$, so solving gives $x_3 = y^2$.

Putting the above together, we get

$$\begin{aligned} P(Y \leq y) &= P\left((X \leq y-2) \cup (-\sqrt{y} \leq x \leq y^2)\right) \\ &= \int_{-3}^{y-2} \frac{1}{4} dx + \int_{-\sqrt{y}}^{y^2} \frac{1}{4} dx \\ &= \frac{1}{4}\left(y + 1 + y^2 + \sqrt{y}\right), \end{aligned}$$

where the $1/4$ comes from the fact that X is uniformly distributed on an interval of length 4.

Combining all regions above gives the CDF

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ \frac{y^2+y+\sqrt{y}+1}{4} & 0 \leq y < 1 \\ 1 & y \geq 1. \end{cases}$$

As a sanity check, it can be good to test the end points. For example, $F_Y(0) = 1/4$, which makes sense because $P(Y = 0) = P(X \leq -2) = 1/4$. Now that we have a CDF we believe in, we can find the PDF through differentiation

$$f_Y(y) = \frac{1}{4}\left(2y + 1 + \frac{1}{2\sqrt{y}}\right).$$