EE 520: Random Processes **Fall 2021**

Exercises 11

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The below are in-class exercises designed to help solidify your understanding of the material covered in the notes. They will also aid you in completing some homework problems. Please work together with your group to complete as many of these problems as you can.

PN refers to the online textbook by Pishro-Nik available here. Please do not look at the solutions until after you have completed the problem or received hints from me.

Exercise 1

For $x, y \in \mathbb{R}^n$, the inner product/dot product is defined as $x^T y = \langle x, y \rangle = \sum_{i=1}^n x_i y_i$. Show that $x^T y = y^T x$.

Exercise 2

Prove that matrix-vector multiplication is a linear operation, i.e., show that the operation

$$f(x) = Ax$$

where $A \in \mathbb{R}^{m \times n}$ and $x \in \mathbb{R}^n$ satisfies $f(x + y) = f(x) + f(y)$ and $f(\alpha x) = \alpha f(x)$.

Exercise 3

Let $A \in \mathbb{R}^{n \times n}$ be diagonal. Write the vector Ax out explicitly.

Exercise 4

Let $A \in \mathbb{R}^{m \times n}$ and $v \in \mathbb{R}^n$. Where must parentheses be placed to make $Av^T v$ a valid operation?

Exercise 5

In the decorrelation example from the lecture notes, how can we make $\mathbb{E}[YY^T] = I$? When this property holds, we say that Y is *isotropic*, and the transformation procedure is called *whitening*.

Exercise 6

Let $X \in \mathbb{R}^n$ be isotropic. Prove that

$$\mathbb{E}\left[\|X\|^2\right] = n.$$

Exercise 7

Let $X, Y \in \mathbb{R}^n$ be independent and isotropic. Prove that

$$\mathbb{E}\left[\langle X, Y \rangle^2\right] = n.$$

Exercise 8

Let $X_1, \ldots, X_n \in \mathbb{R}^p$ be i.i.d. and zero-mean. Define $X \in \mathbb{R}^{p \times n}$ as

$$X = \begin{bmatrix} | & | & & | \\ X_1 & X_2 & \ldots & X_n \\ | & | & & | \end{bmatrix},$$

where the lines above each X_i are a reminder that the X_i's are column vectors. Show that

$$\mathbb{E}\left[\frac{1}{n} X X^T\right] = C_X.$$

This is why we called the the outer product form of matrix multiplication (Sec. 1.3, View 2) the "sample covariance" form.