Ex 1

First consider the probability of 5 arrivals in the window $[0, 60]$.

$$P(N_{60} = 5) = P(N_{60} - N_0 = 5) = \frac{(60\lambda)^5 e^{-60\lambda}}{5!}$$

For the second minute, note that

$$P(N_{t+60} - N_t = 5) = P(N_{60} - N_0 = 5) = P(N_{60} = 5)$$

Therefore

$$P(N_{60} - N_0 = 5 \wedge N_{120} - N_{60} = 5) = P(N_{60} = 5)^2$$

Ex 2

We are interested in $\mathbb{E}[T_{21}]$. Note that

$$X_{21} = T_{21} - T_{20} \sim \exp(\lambda)$$

$$\Rightarrow \mathbb{E}[X_{21}] = \frac{1}{\lambda} = 6 \text{ sec}$$

Therefore

no longer random

$$\mathbb{E}[T_{21}] = \mathbb{E}[X_{21}] + T_{20}$$

$$= 9:01:06 \text{ AM}$$

Ex 3

Note that

$$P(N_{t+T} - N_t = k) = P(N_T - N_0 = k) = P(N_T = k).$$

Now use the law of total probability and independence.

$$P(N_T = k) = \int_{-\infty}^{\infty} P(N_T = k \mid T = t) P(T = t)\, dt$$

$$= \int_0^{\infty} P(N_t = k) P(T = t)\, dt$$

$$= \int_0^{\infty} \frac{(\lambda t)^k}{k!} e^{-\lambda t} \beta e^{-\beta t}\, dt$$

$$= \frac{\beta \lambda^k}{(\lambda + \beta)^{k+1}}$$

Ex 4

$$P\left(N_t^1 = m, N_t^2 = k \mid N_t = m+k\right) = \binom{m+k}{m} p^m (1-p)^k$$

Therefore

$$P(N_t^1 = m, N_t^2 = k) = P\left(N_t^1 = m, N_t^2 = k \mid N_t = m+k\right) P(N_t = m+k)$$

$$= \binom{m+k}{m} p^m (1-p)^k \frac{(\lambda t)^{m+k} e^{-\lambda t}}{(m+k)!}$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$= \frac{p^m (1-p)^k (\lambda t)^{m+k} e^{-\lambda t}}{m!\, k!}$$

$$= \frac{(p\lambda t)^m e^{-p\lambda t}}{m!} \frac{((1-p)\lambda t)^k e^{-(1-p)\lambda t}}{k!} = P(N_t^1 = m) P(N_t^2 = k)$$

Thus, we conclude that N_t^1 is a Poisson RP with rate $p\lambda$ and N_t^2 is Poisson with rate $(1-p)\lambda$. Further, the subprocesses are independent.