Ex 1

Use the definition of inner product:

$$x^T y = \sum_{i=1}^{n} x_i y_i$$
$$= \sum_{i=1}^{n} y_i x_i = y^T x$$

Ex 2

To prove linearity, show the two properties listed.

$$f(x+y) = A(x+y) = Ax + Ay = f(x) + f(y)$$

$$f(\alpha x) = A(\alpha x) = \alpha A x = \alpha f(x)$$

Ex 3

First notice that

$$A = \begin{bmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{bmatrix}.$$

Now think of Ax in terms of the inner products between the rows of A and x, as in Ex 3 above

inner product between vector

$$Ax = \begin{bmatrix} a_{11} & \text{---} & & \\ \text{---} & a_{22} & \text{---} & \\ & & \ddots & \\ \text{---} & & & a_{nn} \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix} = \begin{bmatrix} [a_{11} \; 0 \cdots 0]\, x \\ [0 \; a_{22} \cdots 0]\, x \\ \vdots \\ [0 \; 0 \cdots a_{nn}]\, x \end{bmatrix} \quad \begin{bmatrix} a_{11} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \text{ and } x.$$

For the i^{th} column all but element i will be zero, so we get $a_{ii} x_i$. In general, we get

$$Ax = \begin{bmatrix} a_{11} x_1 \\ a_{22} x_2 \\ \vdots \\ a_{nn} x_n \end{bmatrix}.$$

Ex 4

Sizes are

$A : m \times n$

$v : n \times 1$ $\Rightarrow$ Av^T is <u>not</u> a valid operation but $v^T v$ is

$v^T : 1 \times n$

Therefore we must do $A(v^T v)$ (matlab does this automatically)

Ex 5

We want $\mathbb{E}[YY^T] = I$. With our previous choice $Y = V^T X$, we had $\mathbb{E}[YY^T] = \Lambda$, so in some sense we want to "undo" the Λ matrix. Note that

$$\Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \ddots & & \\ & & \lambda_r & \\ & & & 0 \end{bmatrix}$$

is a diagonal matrix with $\lambda_i \geq 0$, so we can define

$$\Lambda^{-1/2} = \begin{bmatrix} 1/\sqrt{\lambda_1} & & & & \\ & 1/\sqrt{\lambda_2} & & & \\ & & \ddots & & \\ & & & 1/\sqrt{\lambda_r} & \\ & & & & 0 \end{bmatrix}$$

which gives $\Lambda^{-1/2} \Lambda \Lambda^{-1/2} = I$. Hence setting $Y = \Lambda^{-1/2} V^T X$ achieves $\mathbb{E}[YY^T] = I$.

Ex 6

This problem uses a few nice tricks from linear algebra.

$$\mathbb{E}\left[\|X\|^2\right] = \mathbb{E}\left[X^T X\right] \qquad \text{definition of } \|X\|^2$$

$$= \mathbb{E}\left[\operatorname{tr}(X^T X)\right] \qquad \text{trace of scalar}$$

$$= \mathbb{E}\left[\operatorname{tr}(X X^T)\right] \qquad \text{cyclic permutation}$$

$$= \operatorname{tr}\left(\mathbb{E}[X X^T]\right) \qquad \text{linearity of trace}$$

$$= \operatorname{tr}(I_n) = n \qquad \text{isotropy}$$

Ex 7

We use similar tricks to the above plus the law of total expectation.

$$\mathbb{E}[\langle X,Y\rangle^2] = \mathbb{E}_Y\left[\mathbb{E}_{X|Y}\left[\langle X,Y\rangle^2\right]\right]$$

$$= \mathbb{E}_Y\left(\mathbb{E}_{X|Y}\left[Y^T X X^T Y\right]\right)$$

$$= \mathbb{E}_Y\left[Y^T \mathbb{E}_X\left(X X^T\right) Y\right]$$

by the substitution law and the fact that X, Y are independent. Now

$$\mathbb{E}[\langle X,Y\rangle^2] = \mathbb{E}_Y\left[Y^T I Y\right]$$

$$= \mathbb{E}_Y\left(Y^T Y\right)$$

$$= \mathbb{E}_Y\left[\mathrm{tr}\left(Y^T Y\right)\right]$$

$$= \mathbb{E}_Y\left[\mathrm{tr}\left(Y Y^T\right)\right]$$

$$= \mathrm{tr}\left(\mathbb{E}_Y\left[Y Y^T\right]\right)$$

$$= \mathrm{tr}\left(I_n\right) = n$$

Ex 8

$$\mathbb{E}\left[\frac{1}{n} X X^T\right] = \frac{1}{n}\mathbb{E}\left[X X^T\right]$$

$$= \frac{1}{n}\mathbb{E}\left[\sum_{i=1}^{n} X_i X_i^T\right]$$

$$= \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left[X_i X_i^T\right]$$

$$= \frac{1}{n}\cdot n\, C_X = C_X$$