<u>Ex 1-2</u>: See PN.

<u>Ex 3</u>:

a) $$f_X(x) = \int_{y=-\infty}^{\infty} f_{XY}(x,y)\, dy$$

$$= \int_{y=0}^{\infty} e^{-xy}\, dy = -\frac{1}{x} e^{-xy} \Big|_0^{\infty} = \frac{1}{x} \quad \text{for } x \in [1,e]$$

$$f_Y(y) = \int_{x=-\infty}^{\infty} f_{XY}(x,y)\, dx$$

$$= \int_1^e e^{-xy}\, dx = \frac{1}{y}\left(e^{-y} - e^{-ey}\right) \quad \text{for } y > 0$$

b) $$P(0 \le Y \le 1,\ 1 \le X \le \sqrt{e}) = \int_{x=1}^{\sqrt{e}} \int_{y=0}^{1} e^{-xy}\, dy\, dx$$

<u>Ex 4</u>: See PN.

Ex 5

$$\mathbb{E}[X] = \mathbb{E}[X \mid H]\,P(H) + \mathbb{E}[X \mid T]\,P(T)$$

$$= 1 \cdot p + (\mathbb{E}[X] + 1)(1-p)$$

$$= (1-p)\,\mathbb{E}[X] + 1$$

$$\Leftrightarrow p\,\mathbb{E}[X] = 1 \quad \Leftrightarrow \mathbb{E}[X] = \frac{1}{p}$$

Ex 6

The inverse functions h_1 and h_2 are the polar - Cartesian transformation functions

$$h_1(r, \theta) = r \cos \theta$$

$$h_2(r, \theta) = r \sin \theta$$

Applying the formula given above, we get

$$f_{R\Theta}(r, \theta) = f_{X_1 X_2}\big(h_1(y_1, y_2), h_2(y_1, y_2)\big) \, |\det J|$$

$$= f_{X_1 X_2}(r \cos \theta, r \sin \theta) \, r$$

$$= f_{X_1}(r \cos \theta) \, f_{X_2}(r \sin \theta) \, r$$

$$= \underbrace{\frac{1}{2\pi}}_{\text{unif}((-\pi, \pi])} \underbrace{r e^{-r^2/2}}_{\text{Rayleigh}} \quad \text{for } r \geq 0, \ -\pi < \theta \leq \pi$$

Note: R and Θ are independent (you should check by finding marginals)

Ex 7

Yet another method for handling transformations of RVs is to condition, i.e., use the law of total probability.

$$F_Z(z) = P(Z \le z) = \int_{-\infty}^{\infty} P(Z \le z \mid X = x)\, f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} P(e^X Y \le z \mid X = x)\, f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} P(Y \le z e^{-x} \mid X = x)\, f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} F_{Y|X}(z e^{-x} \mid x)\, f_X(x)\, dx$$

Differentiating with respect to Z, we obtain

$$f_Z(z) = \int_{-\infty}^{\infty} f_{Y|X}(z e^{-x} \mid x)\, e^{-x} f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} f_{XY}(x, z e^{-x})\, e^{-x}\, dx$$

Since

$$f_{Y|X}(b \mid a) = \frac{f_{XY}(a, b)}{f_X(a)}.$$

Ex 8

Look at the CDF.

min$(X_1, X_2) \le z_1 \Rightarrow$ at least one $\le z_1$

max$(X_1, X_2) \le z_2 \Rightarrow$ both $\le z_2$

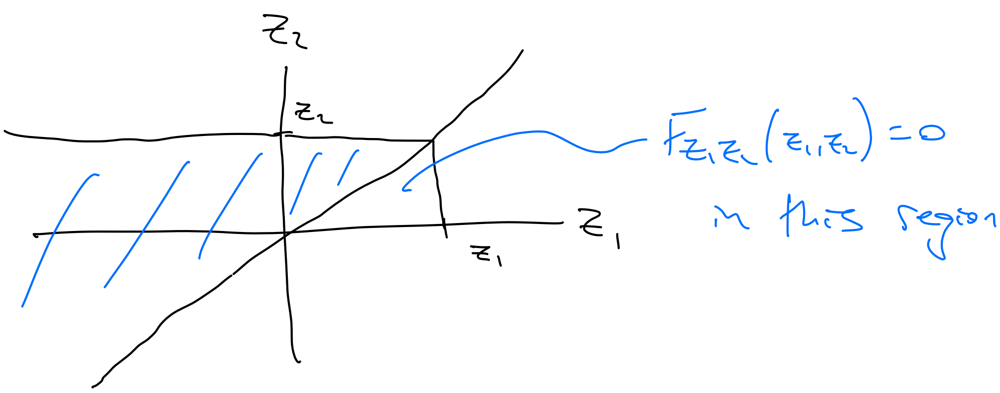
$$F_{Z_1 Z_2}(z_1, z_2) = P\big((Z_1 \le z_1) \cap (Z_2 \le z_2)\big)$$

$$= P\Big(\big(\underbrace{(X_1 \le z_1)}_{A} \cup \underbrace{(X_2 \le z_1)}_{B}\big) \cap \big(\underbrace{(X_1 \le z_2) \cap (X_2 \le z_2)}_{C}\big)\Big)$$

$$= P\big((A \cap C) \cup (B \cap C)\big)$$

$$= P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)$$

Note that min$(X_1, X_2) \le$ max(X_1, X_2), so we must have $Z_1 \le Z_2$.



$$P(A \cap C) = P((X_1 \le z_1) \cap (X_1 \le z_2) \cap (X_2 \le z_2))$$

Since $z_1 \le z_2$, it follows that $X_1 \le z_1 \Rightarrow X_1 \le z_2$. Hence

$$P(A \cap C) = P((X_1 \le z_1) \cap (X_2 \le z_2))$$

$$= F_{X_1 X_2}(z_1, z_2) = F_X(z_1) F_X(z_2)$$

Similarly, since X_1 and X_2 have the same distribution,

$$P(B \cap C) = F_X(z_1) F_X(z_2)$$

Finally

$$P(A \cap B \cap C) = P((X_1 \le z_1) \cap (X_2 \le z_1) \cap (X_1 \le z_2) \cap (X_2 \le z_2))$$

$$= P((X_1 \le z_1) \cap (X_2 \le z_1))$$

$$= F_X^2(z_1)$$

Therefore

$$F_{Z_1 Z_2}(z_1, z_2) = 2 F_X(z_1) F_X(z_2) - F_X^2(z_1) \quad \text{for } z_1 \le z_2.$$

Differentiating gives

$$f_{Z_1 Z_2}(z_1, z_2) = 2 f_X(z_1) f_X(z_2) \quad \text{for } z_1 \le z_2.$$