Ex 1-9: See PN.

Ex 10:

The sample space Ω is the number of possible hands, so $|\Omega| = \binom{52}{14}$. There are 13 cards of each suit, so there are $\binom{13}{2}$ ways to choose 2 spades. Similar reasoning gives $|E| = \binom{13}{2}\binom{13}{3}\binom{13}{4}\binom{13}{5}$. Therefore

$$P(E) = \frac{\binom{13}{2}\binom{13}{3}\binom{13}{4}\binom{13}{5}}{\binom{52}{14}}$$

Ex 11:

There are 100! ways to re-order the packets, so $|\Omega| = 100!$.
For the 10th to be a header, the first 9 must be data.
There are 96 ways to choose the first packet, 95 to choose the second, down to 88 for the ninth. For the tenth packet, we have 4 options. We then have 90! ways to choose the last 90 packets. This gives

$$|E| = 96 \cdot 95 \cdot \ldots \cdot 88 \cdot 4 \cdot 90!$$

$$\Rightarrow P(E) = \frac{96 \cdot 95 \cdot \ldots \cdot 88 \cdot 4 \cdot 90!}{100!}$$

$$= \frac{96 \cdot \ldots \cdot 88 \cdot 4}{100 \cdot \ldots \cdot 91}$$

Ex 12:

a) In this case, $X_1 \sim Ber(\frac{1}{2})$, since there are two balls in the urn. Similarly, since we put the first ball back in, $X_2 \sim Ber(\frac{1}{2})$, so they are identically distributed. Are they independent? We need to check that

$$P(X_1 = x_1, X_2 = x_2) = P(X_1 = x_1) P(X_2 = x_2).$$

This is easy to verify by noting that all four outcomes 00, 01, 10, 11 are equally likely.

(b) In this case, we have to be careful not to confuse the marginal probability $P(X_2)$ with the conditional probability $P(X_2 | X_1)$. Clearly $P(X_2 | X_1) = 1$ for whichever ball was not drawn first, and hence X_1 and X_2 are not independent. However,

$$P(X_2 = 1) = P(X_2 = 1 | X_1 = 0) P(X_1 = 0) + P(X_2 = 1 | X_1 = 1) P(X_1 = 1)$$

$$= 1 \cdot \frac{1}{2} = \frac{1}{2}.$$

Thus X_1 and X_2 are both $Ber(\frac{1}{2})$, i.e., X_1 and X_2 are identically distributed but <u>not</u> independent.

Ex 13:

(a) Define the indicator RV

$$X_i = \begin{cases} 1 & i\text{th toss is a new run} \\ 0 & \text{otherwise} \end{cases}$$

Clearly $X_1 = 1$ always. Now for X_i with $i \geq 2$, we start a new run if toss i is not equal to toss $i-1$. This happens with probability $\frac{1}{2}$, so

$$P(X_i = 1) = \frac{1}{2} \quad \text{for } i \geq 2.$$

Therefore

$$\mathbb{E}[\#\text{ runs}] = 1 + \sum_{i=2}^{n} P(X_i = 1)$$

$$= 1 + \frac{1}{2}(n-1)$$

(b) In this case, we consider the probability of a new run conditioned on the previous toss. Let T_i be the RV denoting the result of the i^{th} toss. Then $X_i = 1$ if $T_i \neq T_{i-1}$. Therefore

$$P(X_i = 1) = P(T_i \neq T_{i-1})$$

$$= P(T_i \neq T_{i-1} \mid T_i = H)\, P(T_i = H) + P(T_i \neq T_{i-1} \mid T_i = T)\, P(T_i = T)$$

$$= (1-p)p + p(1-p) = 2p(1-p)$$

Thus

$$E[\#\text{ runs}] = 1 + 2(n-1)p(1-p)$$