EE 510: Mathematical Foundations of Machine Learning **Winter 2020**

Exercises 5

Pages: 3.21 - 3.46 *Instructor Name: John Lipor*

Exercise 1

Let $A \in \mathbb{R}^{m \times n}$ with $m \geq n$. Give an intuitive description of what it means to have $\text{rank}(A) < n$.

Exercise 2

Show that for $A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{n \times p}$, every column of $B^T A^T$ is a linear combination of the rows of B.

Exercise 3

Prove that $\text{rank}\left(\prod_{k=1}^{K} A_k\right) \leq \min_k \{\text{rank}(A_k)\}$.

Exercise 4

Let $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$ and $A = xy^T$. What is rank(A)? What is it when x and y are orthogonal (assuming $m = n$)?

Exercise 5

Intuitively, why does multiplying by a unitary matrix have no impact on the rank?

Exercise 6

Let $Q \in \mathbb{R}^{n \times n}$ be orthogonal. Prove $\text{rank}(AQ) = \text{rank}(A)$.

Exercise 7

Let $A \in \mathbb{R}^{m \times n}$. What size is $y \in \mathcal{R}^{\perp}(A)$? What size is $x \in \mathcal{N}(A)$? What is true of $x^T y$ in this case?

Exercise 8

- Let $x_0 \in \mathcal{N}(A)$, $x_1 \in \mathcal{N}^{\perp}(A)$. What is $x_0^T x_1$?
- Let $y_1 \in \mathcal{R}(A)$, $y_0 \in \mathcal{R}^{\perp}(A)$. What is $y_0^T y_1$?

Exercise 9

Show that $\mathcal{R}^{\perp}(A) = \mathcal{N}(A^T)$ following the similar proof on 3.29.

Exercise 10

What are the bases for the four fundamental subspaces in terms of the SVD? What are the dimensions of these matrices?

Exercise 11

Let $A \in \mathbb{R}^{m\times n}$. What must be true to have $\text{range}(A) = \mathbb{R}^m$?

Exercise 12

What is the difference between the full and thin SVD? Are both always valid?

Exercise 13

Let U be the matrix of left singular vectors and U_r be the first r columns of U (e.g., from the thin SVD). What are UU^T, U^TU, $U_rU_r^T$, and $U_r^TU_r$?

Exercise 14

Let $U_r \in \mathbb{R}^{m\times r}$ have orthonormal *columns* (i.e., U_r could be a matrix of left singular vectors from the thin SVD of some arbitrary matrix). For a given vector $x \in \mathbb{R}^m$ (unrelated to the matrix that gave us U_r), how are $\left\|U_r^T x\right\|_2$ and $\left\|U_r U_r^T x\right\|_2$ related?

Exercise 15

Let $A = zz^T$ for $z \in \mathbb{R}^n$. Write the synthesis view of the eigendecomposition of A.

Exercise 16

Verify the eigendecomposition on pg. 3.43 is valid. Verify the SVD on pg. 3.44 is valid.

Exercise 17

Let $y = Ax$. Write the coordinates of y in terms of the basis $U_r \in \mathbb{R}^{m\times r}$ of left singular vectors of the rank(r) matrix A.