Ex 1

$$\|B-A\|_F^2 = tr\left((B-A)^T(B-A)\right)$$

$$= tr\left((B^T-A^T)(B-A)\right)$$

$$= tr\left(B^TB + A^TA - A^TB - B^TA\right)$$

$$= tr(B^TB) + tr(A^TA) - 2\,tr(A^TB)$$

Ex 2

Consider $X = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $Y = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Take $t = \frac{1}{2}$. Then

$$(1-t)X + tY = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

which has rank 2, so it's not in the set of rank-1 matrices.

Ex 3

$$\|\hat{B}-A\|_F^2 = \left\|\sum_{k=1}^{K} \sigma_k u_k v_k^T - \sum_{k=1}^{r} \sigma_k u_k v_k^T\right\|_F^2$$

$$= \left\|\sum_{k=1}^{k} (\sigma_k u_k v_k^T - \sigma_k u_k v_k^T) + \sum_{k=K+1}^{r} \sigma_k u_k v_k^T\right\|_F^2$$

$$= \left\|\sum_{k=K+1}^{r} \sigma_k u_k v_k^T\right\|_F^2$$

$$= \|\bar{U}\bar{\Sigma}\bar{V}^T\|_F^2$$

where $\bar{U} = U[:, K+1:r]$

$\bar{V} = V[:, K+1:r]$

$\bar{\Sigma} = \Sigma[K+1:r, K+1:r]$

$$= tr\left((\bar{U}\bar{\Sigma}\bar{V}^T)^T(\bar{U}\bar{\Sigma}\bar{V}^T)\right)$$

$$= tr\left(\bar{V}\bar{\Sigma}^T\bar{U}^T\bar{U}\bar{\Sigma}\bar{V}^T\right)$$

$\bar{U}^T\bar{U} = I_{r-K}$

$$= tr\left(\bar{V}^T\bar{V}\bar{\Sigma}^T\bar{\Sigma}\right)$$

cyclic permutation property of trace

$$= tr\left(\bar{\Sigma}^T\bar{\Sigma}\right)$$

$$= \sum_{k=K+1}^{r} \sigma_k^2$$

Ex 4

$$U^T \left(\sum_{k=K+1}^{r} \sigma_k u_k v_k^T \right) = \sum_{k=K+1}^{r} \sigma_k \left(U^T u_k \right) v_k^T$$

Now note that $U^T u_k = e_k$, so

$$\sum_{k=K+1}^{r} \sigma_k \left(U^T u_k \right) v_k^T = \sum_{k=K+1}^{r} \sigma_k e_k v_k^T$$

Now right multiply by V and note that $v_k^T V = e_k^T$ to see that

$$U^T \left(\sum_{k=K+1}^{r} \sigma_k u_k v_k^T \right) V = \sum_{k=K+1}^{r} \sigma_k e_k e_k^T .$$

Ex 5

See pg. 6.15.

Ex 6

A basis must span the corresponding vector space and consist of LI vectors, so we need to show that

1) $e_m e_n^T$ spans $\mathbb{R}^{m \times n}$

2) the vectors $\{e_m e_n^T : m=1,\dots,M,\ n=1,\dots,N\}$ are LI

Note that $e_m e_n^T$ is an $M \times N$ matrix E with $E_{mn} = 1$ and zero everywhere else. Therefore any $A \in \mathbb{R}^{m \times n}$ can be written as

$$A = \sum_{m=1}^{M} \sum_{n=1}^{N} a_{mn} e_m e_n^T$$

meaning the vectors span the space. In Ex. 8, we'll show these vectors are orthogonal, implying they are LI.

Ex 7

Let $x = A$, $y = B$. Then we have

- $\langle x, y \rangle = tr(B^T A) = tr((A^T B)^T) = tr(A^T B) = \langle y, x \rangle$

- $\langle \alpha x, y \rangle = tr(\alpha A^T B) = \alpha\, tr(A^T B) = \alpha \langle x, y \rangle$

- $\langle x+y, z \rangle = tr((A+B)^T C) = tr((A^T + B^T) C) = tr(A^T C + B^T C)$
  $= tr(A^T C) + tr(B^T C) = \langle x, z \rangle + \langle y, z \rangle$

- $\langle x, x \rangle = tr(A^T A) = \sum_{i=1}^{n} \lambda_i$ where λ_i are the eigenvalues of $A^T A$
  but $A^T A$ is PSD, so $\lambda_i \geq 0$ and all are zero $\iff A = 0$.

Ex 8

Yes. Following pg. 6.18,

$$\langle e_m e_n^T, e_k e_\ell^T \rangle = tr\left((e_k e_\ell^T)^T (e_m e_n^T)\right) = tr\left(e_\ell e_k^T e_m e_n^T\right)$$

$$= (e_k^T e_m)(e_n^T e_\ell) = \begin{cases} 1 & k=m,\ \ell=n \\ 0 & \text{otherwise} \end{cases}$$

Ex 9

Read it!

Ex 10

Let $\mathbb{1} = [1\ 1\ \cdots\ 1]^T \in \mathbb{R}^n$

$$(P^\perp)^2 = \left(I - \frac{1}{n}\mathbb{1}\mathbb{1}^T\right)\left(I - \frac{1}{n}\mathbb{1}\mathbb{1}^T\right)$$

$$= I - \frac{1}{n} I \mathbb{1}\mathbb{1}^T - \frac{1}{n}\mathbb{1}\mathbb{1}^T I + \frac{1}{n^2}\mathbb{1}\mathbb{1}^T\mathbb{1}\mathbb{1}^T$$

note for $\mathbb{1} \in \mathbb{R}^n$,

$\mathbb{1}^T\mathbb{1} = \sum_{i=1}^{n} 1 = n$

$$= I - 2\,\frac{1}{n}\mathbb{1}\mathbb{1}^T + \frac{1}{n}\mathbb{1}\mathbb{1}^T$$

$$= I - \frac{1}{n}\mathbb{1}\mathbb{1}^T$$

Ex 11

$$P^{\perp}\mathbb{1} = \left(I - \frac{1}{n}\mathbb{1}\mathbb{1}^T\right)\mathbb{1}$$

$$= \mathbb{1} - \frac{1}{n}\mathbb{1}\mathbb{1}^T\mathbb{1}$$

$$= \mathbb{1} - \frac{1}{n}\mathbb{1}\,n$$

$$= \mathbb{1} - \mathbb{1} = 0$$