Ex 1

First consider $\mathbb{R}$ and verify the definition. If $x, y \in \mathbb{R}$, then $x+y \in \mathbb{R}$. Also if $x \in \mathbb{R}$ and $\alpha \in \mathbb{F}$ (which we take to also be $\mathbb{R}$), then $\alpha x \in \mathbb{R}$.

Next, the $y=0$ axis in $\mathbb{R}^2$ is all vectors of the form $\begin{bmatrix} x_1 \\ 0 \end{bmatrix}$. Let $u = \begin{bmatrix} u_1 \\ 0 \end{bmatrix}$, $v = \begin{bmatrix} v_1 \\ 0 \end{bmatrix}$, $\alpha \in \mathbb{R}$. Then

$$u + v = \begin{bmatrix} u_1 + v_1 \\ 0 \end{bmatrix}$$

which is on the $y=0$ axis. Further $\alpha u = \begin{bmatrix} \alpha u_1 \\ 0 \end{bmatrix}$ also is.

Ex 2

The "linear combination of columns" view of matrix-vector multiplication is most appropriate. Matrix-matrix multiplication versions 2 and 3 are both appropriate.

Ex 3

$$\mathrm{span}(\{u_1, \dots, u_n\}) = \left\{ y = \sum_{i=1}^{n} \alpha_i u_i : \alpha_i \in \mathbb{F} \right\}$$

Now take two vectors $x, y \in \mathrm{span}(\{u_1, \dots, u_n\})$, and $\gamma \in \mathbb{F}$. Then we can write

$$x = \sum_{i=1}^{n} \alpha_i u_i , \quad y = \sum_{i=1}^{n} \beta_i u_i$$

for some $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n$. Therefore

$$x + y = \sum_{i=1}^{n} \alpha_i u_i + \sum_{i=1}^{n} \beta_i u_i$$

$$= \sum_{i=1}^{n} \underbrace{(\alpha_i + \beta_i)}_{\text{call this } \delta_i} u_i = \sum_{i=1}^{n} \delta_i u_i$$

So $x+y \in \mathrm{span}(\{u_1, \dots, u_n\})$. Further,

$$\gamma x = \gamma \sum_{i=1}^{n} \alpha_i u_i$$

$$= \sum_{i=1}^{n} \gamma \alpha_i u_i$$

$$= \sum_{i=1}^{n} \underbrace{(\gamma \alpha_i)}_{\xi_i} u_i = \sum_{i=1}^{n} \xi_i u_i$$

So $\gamma x \in \mathrm{span}(\{u_1, \dots, u_n\})$, and it is therefore a subspace.

Ex 4

Let $u \in \mathbb{R}^3$ be a vector on the (x,z) plane. Any such vector has the form $u = \begin{bmatrix} u_1 \\ 0 \\ u_3 \end{bmatrix}$. We can write any such vector as

$$u = \frac{1}{2}(u_1 + u_3)\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{2}(u_1 - u_3)\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Since u was arbitrary, the vectors span the (x,z) plane.

Ex 5

a) Yes. If not, we could write some c_j as

$$c_j = \begin{bmatrix} a_j \\ b_j \end{bmatrix} = \sum_{i=1}^{k} \alpha_i \begin{bmatrix} a_i \\ b_i \end{bmatrix}$$

but this would require $a_j = \sum_{i=1}^{k} \alpha_i a_i$, contradicting the linear independence of the a_i's.

b) Not necessarily. If $b_1, \dots, b_k$ are linearly independent, then so are $c_1, \dots, c_k$. For example, let $a_1, \dots, a_k = 1$ $(a_i \in \mathbb{R})$. Then

$$c_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } c_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$ are linearly independent.

Ex 6

Let $x, y \in \mathbb{R}^n$ be orthogonal, and suppose they are linearly dependent. Then we can write $x = \alpha y$ for some $\alpha \in \mathbb{R}$, and so

$$x^T y = (\alpha y)^T y = \alpha y^T y = \alpha \|y\|.$$

Assuming $y \neq 0$, $\|y\| > 0$, contradicting the orthogonality of x and y.

Ex 7

Suppose they are linearly dependent, so that $\exists \beta_1, \beta_2, \beta_3 \in \mathbb{R}$ such that $\beta_1 a_1 + \beta_2 a_2 + \beta_3 a_3 = 0$. This would mean

$$\beta_1 - \beta_3 = 0 \qquad -\beta_2 + \beta_3 = 0 \qquad \beta_2 + \beta_3 = 0$$

The last two imply $2\beta_3 = -0$, so $\beta_3 = 0$, which implies $\beta_1 = 0$ and $\beta_2 = 0$. Therefore, the only way to have $\beta_1 a_1 + \beta_2 a_2 + \beta_3 a_3 = 0$ is if $\beta_1 = \beta_2 = \beta_3 = 0$, from which we conclude that a_1, a_2, a_3 are linearly independent.

Ex 8

$$\|x\|^2 = x^T x = \left(\sum_{i=1}^{k} \beta_i a_i\right)^T \left(\sum_{j=1}^{k} \beta_j a_j\right)$$

$$= \sum_{i=1}^{k} \beta_i^2 \underbrace{a_i^T a_i}_{1} + \sum_{\substack{i=1 \\ j \neq i}}^{k} \beta_i \beta_j \underbrace{a_i^T a_j}_{0}$$

$$= \sum_{i=1}^{k} \beta_i^2 = \|\beta\|^2$$

Therefore $\|x\| = \|\beta\|$.

Ex 9

- First, it is obvious that the vectors are linearly independent. Now to show they span $\mathbb{R}^n$, note that for any $x \in \mathbb{R}^n$,

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n .$$

- The vectors are already shown to be linearly independent. We could simply note that $\dim(\mathbb{R}^3) = 3$, so the vectors must span $\mathbb{R}^3$. More directly, for $x \in \mathbb{R}^3$,

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \left(- \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)$$

$$+ x_3 \left(\frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right)$$

$$= a_1 \left(x_1 + x_2 + \frac{1}{2} x_3 \right) + a_2 \left(-x_2 + \frac{1}{2} x_3 \right) + a_3 \left(x_2 + \frac{1}{2} x_3 \right)$$

Ex 10

Among $u_1, \dots, u_N$, the maximum number of linearly independent vectors is N, so the maximum number of basis vectors for $\mathrm{span}(\{u_1, \dots, u_N\})$ is N.

Ex 11

- $\dim(S) = 1$
- $S + T = \left\{ s + t : s \in \mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right),\ t \in \mathrm{span}\left(\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}\right) \right\}$

These are all vectors of the form $\begin{bmatrix} a \\ 0 \\ b \end{bmatrix}$, so $S + T$ is the (x, z) plane, and $\dim(S + T) = 2$.

- $S \cap T = \{ v \in \mathbb{R}^3 : v \in \text{span}\left(\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}\right\}\right), v \in \text{span}\left(\left\{\begin{bmatrix}1\\0\\-1\end{bmatrix}\right\}\right)$

  Note that these vectors are orthogonal, so $S \cap T = \{0\}$.

  This has infinite dimension.

<u>Ex 12</u>

Since all vectors are in $\mathbb{R}^3$, the maximum dimension is 3, in which case the vectors in $S+T$ would span $\mathbb{R}^3$. We need to check if any three are linearly independent. We know that T spans the (x,z) plane, so we need to see if either vector in S is linearly independent of these two. We need to see if

$$\begin{bmatrix}1\\1\\0\end{bmatrix} = \alpha_1 \begin{bmatrix}1\\0\\1\end{bmatrix} + \alpha_2 \begin{bmatrix}1\\0\\-1\end{bmatrix}$$

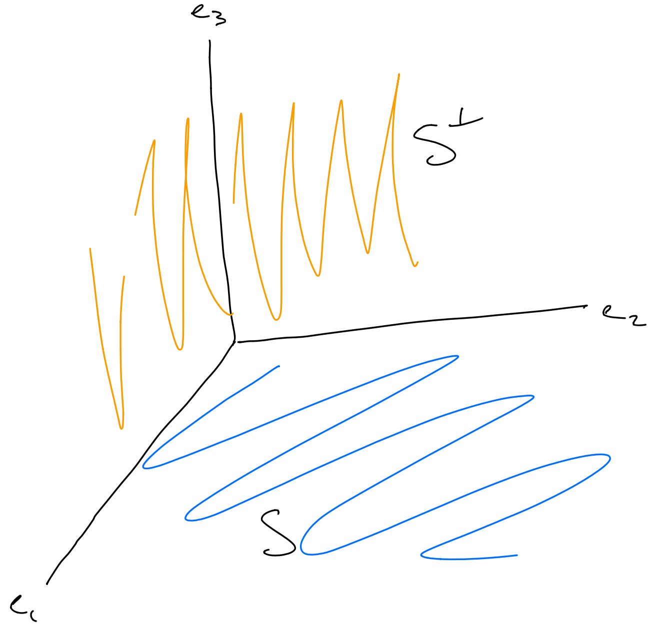
but this is clearly impossible, so $\dim(S+T) = 3$ and $S+T$ spans $\mathbb{R}^3$.

Next note that

$$\begin{bmatrix}1\\1\\0\end{bmatrix} - \begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}1\\0\\-1\end{bmatrix} \in T$$

So $\begin{bmatrix}1\\0\\-1\end{bmatrix} \in S \cap T \Rightarrow S+T$ is <u>NOT</u> a direct sum.

Ex 13



Ex 14

Note that $\mathcal{R}(A) = \{ y = Ax : x \in \mathbb{R}^n \}$, so $\mathcal{R}(A)$ is a subspace of $\mathbb{R}^m$. By definition, $S^\perp$ is the subspace of vectors orthogonal to $\mathcal{R}(A)$. Since we have to be able to compute $\langle v, y \rangle$ for $y \in \mathcal{R}(A)$, we must have $v \in \mathbb{R}^m$ as well.

Therefore, vectors in $S^\perp$ have dimension m.

Ex 15

Since $\mathcal{R}(A) \subset \mathbb{R}^m$, its maximum dimension is m. Recall that

$$\mathcal{R}(A) = \text{span}(\text{columns of } A)$$

and A has $\underline{n}$ columns. Therefore, to have $\dim(\mathcal{R}(A)) = m$, A must have at least m linearly independent columns.