Ex 1

The eigenvalues of A are 13 and 0. First verify

$$Av = \lambda v$$

$$\Leftrightarrow \begin{bmatrix} 4 & 6 \\ 6 & 9 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 26 \\ 39 \end{bmatrix} = 13 \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

so the eigenvector equation is satisfied. However

$$\|v\|^2 = 2^2 + 3^2 = 13 \neq 1$$

so v does not have unit norm and is therefore not a valid eigenvector. Instead we use

$$v = \frac{1}{\sqrt{13}} \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Ex 2

First use the inner product view of matrix-matrix multiplication.

Let the columns of V be $v_1, \dots, v_n \in \mathbb{R}^n$, so that

$$V = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} \quad \text{and} \quad V^T = \begin{bmatrix} - & v_1^T & - \\ - & v_2^T & - \\ & \vdots & \\ - & v_n^T & - \end{bmatrix}$$

Then

$$V^T V = \begin{bmatrix} v_1^T v_1 & v_1^T v_2 & \cdots & v_1^T v_n \\ v_2^T v_1 & v_2^T v_2 & \cdots & v_2^T v_n \\ \vdots & & & \\ v_n^T v_1 & v_n^T v_2 & \cdots & v_n^T v_n \end{bmatrix}$$

Since the columns of V are orthogonal,

$$v_i^T v_j = \begin{cases} 0 & \text{if } i \neq j \\ v_i^T v_i & \text{if } i = j \end{cases}$$

Thus V is diagonal. By the above, the diagonal elements of V are of the form $v_i^T v_i = \|v_i\|^2$. Therefore $V^T V = I$ iff $\|v_i\|^2 = 1$ for $i = 1, \dots, n$, i.e., if the columns are orthonormal.

Ex 3

A matrix is symmetric if $A = A^T$. First let $A = X^T X$.

$$A^T = (X^T X)^T = X^T (X^T)^T = X^T X = A$$

Next let $A = XX^T$. Then

$$A^T = (XX^T)^T = (X^T)^T X^T = XX^T = A$$

Ex 4

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow P^T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$P^T P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$PP^T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So $P^T P = PP^T = I$, which is interesting to note.

Ex 5

$$U^T U = \begin{bmatrix} \cos\theta & \sin\theta \\ -q\sin\theta & q\cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -q\sin\theta \\ \sin\theta & q\cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & -q\sin\theta\cos\theta + q\sin\theta\cos\theta \\ -q\sin\theta\cos\theta + q\sin\theta\cos\theta & q^2(\sin^2\theta + \cos^2\theta) \end{bmatrix}$$

Note that $\cos^2\theta + \sin^2\theta = 1$. Further, since $q \in \{0, 1\}$, $q^2 = 1$, so

$$U^T U = I.$$

Ex 6

We will commonly replace A with its SVD, i.e., $A = U\Sigma V^T$.

$$Ay = A(v_1 + v_2) = U\Sigma V^T (v_1 + v_2)$$

$$= U\Sigma V^T v_1 + U\Sigma V^T v_2$$

$$= U\Sigma (V^T v_1 + V^T v_2)$$

Now note that if v_j is the j^{th} column of V, we have

$$V^T v_j = \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_j^T \\ \vdots \\ v_n^T \end{bmatrix} v_j = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = e_j$$

called the "j^{th} standard basis vector," which is zero everywhere except element j

So

$$Ay = U\Sigma\left(V^T v_1 + V^T v_2\right) = U\Sigma\left(e_1 + e_2\right)$$

Now

$$\Sigma e_j = \begin{array}{c c} & \begin{array}{cc} r & n-r \end{array} \\ \begin{array}{c} r \\ \\ m-r \end{array} & \left[\begin{array}{cc} \begin{matrix} \sigma_1 & & & & \\ & \ddots & & & \\ & & \sigma_j & & \\ & & & \ddots & \\ & & & & \sigma_r \end{matrix} & 0_{r \times n-r} \\ 0_{m-r \times r} & 0_{m-r \times n-r} \end{array}\right] \end{array} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \sigma_j \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

So that

$$U\Sigma e_j = \sigma_j u_j$$ (write it out like above to see this)

Therefore

$$Ay = \sigma_1 u_1 + \sigma_2 u_2.$$

Now note that $u_1 \perp u_2$, so

$$\|Ay\|^2 = \|\sigma_1 u_1 + \sigma_2 u_2\|^2$$

$$= \sigma_1^2 \overbrace{\|u_1\|^2}^{1} + \sigma_2^2 \overbrace{\|u_2\|^2}^{1} + 2\sigma_1\sigma_2 \overbrace{u_1^T u_2}^{0}$$

$$= \sigma_1^2 + \sigma_2^2$$

$$\Rightarrow \|Ay\| = \sqrt{\sigma_1^2 + \sigma_2^2}$$

<u>Ex 7</u>

Similar to pg. 2.18, we have $A^T z = V \Sigma^T U^T z$. We want $A^T z \perp A^T x$, i.e., $(A^T z)^T (A^T x) = 0$. Expanding, we get

$$z^T A^T A x = z^T V \Sigma \overbrace{U^T U}^{I} \Sigma V^T x$$

$$= z^T V \Sigma^T \Sigma V^T x$$

$$= z^T V \begin{bmatrix} \sigma_1^2 & & & \\ & \ddots & & \\ & & \sigma_r^2 & 0 \\ & 0 & & 0 \end{bmatrix} V^T x$$

Therefore taking $z = u_1$ and $x = u_2$ gives

$$z^T A^T A x = (u_1^T V)(\Sigma^T \Sigma)(V^T u_2)$$

$$= e_1^T \Sigma^T \Sigma e_2$$

$$= \sigma_1 \sigma_2 e_1^T e_2 = 0$$

Since $e_1^T e_2 = [1\ 0 \cdots 0] \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = 0$. An easier way to solve this is to recognize that $A^T u_j = \sigma_j u_j$ by the reasoning of the previous exercise. In this case

$$(A^T z)^T (A^T x) = (\sigma_1 u_1)^T (\sigma_2 u_2)$$

$$= \sigma_1 \sigma_2 u_1^T u_2 = 0$$

Since all singular vectors are orthogonal.

Ex 8

See notes.

Ex 9

See notes.

Ex 10

Write X as $U\Sigma V^T$. Then we see that

$$A = X^T X = V\Sigma^T U^T U \Sigma V^T$$
$$= V \Sigma^T \Sigma V^T$$

but $\Sigma^T\Sigma$ is diagonal and V is an orthogonal matrix, so $V(\Sigma^T\Sigma)V^T$ is a valid eigendecomposition. We therefore conclude that the eigenvectors of A are the <u>right</u> singular vectors of X and the eigenvalues are the <u>squares</u> of the singular values of X.

What would happen if we let $A = XX^T$?