Ex 1

Vectorization is done by stacking the <u>columns</u> of a matrix.

$$A = \underbrace{\begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix}}_{m \times n} \rightarrow a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^{mn}$$

Ex 2

To prove linearity, show the two properties listed.

$$f(x+y) = A(x+y) = Ax + Ay = f(x) + f(y)$$

$$f(\alpha x) = A(\alpha x) = \alpha A x = \alpha f(x)$$

Ex 3

Let $X = \begin{bmatrix} x_1^T \\ x_2^T \\ \vdots \\ x_N^T \end{bmatrix} \in \mathbb{R}^{N \times n}$. Then we have

$$Xw = \begin{bmatrix} x_1^T \\ x_2^T \\ \vdots \\ x_N^T \end{bmatrix} w = \begin{bmatrix} x_1^T w \\ x_2^T w \\ \vdots \\ x_N^T w \end{bmatrix} \approx \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}$$

Ex 4

See pages 1.12-1.13.

Ex 5

First notice that

$$A = \begin{bmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{bmatrix}.$$

Now think of Ax in terms of the inner products between the rows of A and x, as in Ex 3 above

$$Ax = \begin{bmatrix} a_{11} & \text{---} & & \\ \text{---} & a_{22} & \text{---} & \\ & & \ddots & \\ \text{---} & & & a_{nn} \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix} = \begin{bmatrix} [a_{11} \; 0 \cdots 0]\, x \\ [0 \; a_{22} \cdots 0]\, x \\ \vdots \\ [0 \; 0 \cdots a_{nn}]\, x \end{bmatrix}$$

inner product between vector $\begin{bmatrix} a_{11} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ and x.

For the i^{th} column all but element i will be zero, so we get $a_{ii} x_i$. In general we get

$$Ax = \begin{bmatrix} a_{11} x_1 \\ a_{22} x_2 \\ \vdots \\ a_{nn} x_n \end{bmatrix}.$$

Ex 6

For a diagonal matrix

$$A = \begin{bmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{bmatrix},$$

we have

$$A^{-1} = \begin{bmatrix} 1/a_{11} & & & \\ & 1/a_{22} & & \\ & & \ddots & \\ & & & 1/a_{nn} \end{bmatrix}.$$

Now check AA^{-1}:

$$AA^{-1} = \begin{bmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{bmatrix} \begin{bmatrix} 1/a_{11} & & & \\ & 1/a_{22} & & \\ & & \ddots & \\ & & & 1/a_{nn} \end{bmatrix}$$

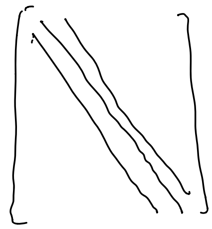
Think of A^{-1} as a series of columns. Then by Ex 5, the first column is

$$A \begin{bmatrix} \frac{1}{a_{11}} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} \cdot \frac{1}{a_{11}} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

By similar reasoning, we get $AA^{-1} = A^{-1}A = I$.

Ex 7

A tridiagonal matrix has the form

$$A = $$  $$= \begin{bmatrix} a_{11} & a_{12} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & a_{23} & 0 & \cdots & 0 \\ 0 & a_{32} & a_{33} & a_{34} & \cdots & 0 \\ & & \ddots & \ddots & \ddots & \\ 0 & 0 & \cdots & & a_{nn-1} & a_{nn} \end{bmatrix}$$

Thus for Ax, each element will only have 2 or 3 nonzero elements in the sum. The elements of $y = Ax$ will be

$$y_1 = a_{11}x_1 + a_{12}x_2$$

$$y_i = a_{ii-1}x_{i-1} + a_{ii}x_i + a_{ii+1}x_{i+1}, \quad i = 2, \ldots, n-1$$

$$y_n = a_{nn-1}x_{n-1} + a_{nn}x_n$$

Ex 8

To see the form of A, write out a few values of y.

$$y_1 = x_1$$
$$y_2 = x_1 + x_2$$
$$y_3 = x_1 + x_2 + x_3$$

So we need a matrix that "picks out" the correct elements of x for each element of y. This has the form

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 0 \\ \vdots & & & & \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 + x_2 \\ x_1 + x_2 + x_3 \\ \vdots \\ x_1 + x_2 + \cdots + x_N \end{bmatrix}$$

Ex 9

This problem is most easily derived by iterative substitution. First let $V_1 = (u_2 u_3 \cdots u_K)$. Then

$$(u_1 u_2 \cdots u_K)^T = (u_1 V_1)^T$$
$$= V_1^T u_1^T \quad \text{by traspose property, pg. 1.14}$$
$$= (u_2 \cdots u_K)^T u_1^T$$

Now we need to evaluate $(U_2 \cdots U_K)^T$. Let $V_2 = (U_3 U_4 \cdots U_K)$.

Then

$$
\begin{aligned}
(U_2 \cdots U_K)^T &= (U_2 V_2)^T \\
&= V_2^T U_2^T \\
&= (U_3 U_4 \cdots U_K) U_2^T,
\end{aligned}
$$

and therefore

$$(U_1 U_2 \cdots U_K)^T = (U_3 U_4 \cdots U_K)^T U_2^T U_1^T.$$

Repeating this process $K-3$ more times shows that

$$(U_1 U_2 \cdots U_K)^T = U_K^T U_{K-1}^T \cdots U_3^T U_2^T U_1^T.$$

The second bullet is then

$$(U_1 U_2 \cdots U_K)(U_1 U_2 \cdots U_K)^T = U_K^T U_{K-1}^T \cdots U_3^T U_2^T U_1^T U_1 U_2 U_3 \cdots U_{K-1} U_K$$

Ex 10

For three matrices, use substitution again. Let $D = B+C$. Then

$$(A+B+C)^T = (A+D)^T$$

$$= A^T + D^T \quad \leftarrow \text{by transpose property for 2 matrices}$$

$$= A^T + (B+C)^T$$

$$= A^T + B^T + C^T$$

To prove this holds for K matrices via induction, we need to show the base case then prove the inductive hypothesis. The base case is $K=2$, which is given to you. For the inductive hypothesis, assume the statement holds for $K-1$ matrices, i.e., that

$$(A_1 + A_2 + \cdots + A_{K-1})^T = A_1^T + A_2^T + \cdots + A_{K-1}^T.$$

We wish to show that

$$(A_1 + A_2 + \cdots + A_{K-1} + A_K)^T = A_1^T + A_2^T + \cdots + A_{K-1}^T + A_K^T.$$

Let $B = A_1 + A_2 + \cdots + A_{K-1}$. Then

$$(A_1 + A_2 + \cdots + A_{K-1} + A_K)^T = (B + A_K)^T \qquad \text{substitution}$$

$$= B^T + A_K^T \qquad \text{using base case}$$

$$= A_1^T + \cdots + A_{K-1}^T + A_K^T \qquad \text{by inductive hypothesis}$$