Chapter 1

What This Book Is About and How to Read It

1.1 "Exercises" vs. "Problems"

This is a book about mathematical problem solving. We make three assumptions about you, our reader:

- You enjoy math.
- You know high-school math pretty well, and have at least begun the study of "higher mathematics" such as calculus and linear algebra.
- You want to become better at solving math problems.

First, what is a **problem**? We distinguish between **problems** and **exercises**. An exercise is a question that you know how to resolve immediately. Whether you get it right or not depends on how expertly you apply specific techniques, but you don't need to puzzle out what techniques to use. In contrast, a problem demands much thought and resourcefulness before the right approach is found. For example, here is an exercise.

Example 1.1.1 Compute 5436^3 without a calculator.

You have no doubt about how to proceed—just multiply, carefully. The next question is more subtle.

Example 1.1.2 Write

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{99\cdot 100}$$

as a fraction in lowest terms.

At first glance, it is another tedious exercise, for you can just carefully add up all 99 terms, and hope that you get the right answer. But a little investigation yields something intriguing. Adding the first few terms and simplifying, we discover that

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}=\frac{2}{3},$$

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}=\frac{3}{4},$$
$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}=\frac{4}{5},$$

which leads to the **conjecture** that for all positive integers n,

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}.$$

So now we are confronted with a **problem**: is this conjecture true, and if so, how do we *prove* that it is true? If we are experienced in such matters, this is still a mere exercise, in the technique of mathematical induction (see page 45). But if we are not experienced, it is a problem, not an exercise. To solve it, we need to spend some time, trying out different approaches. The harder the problem, the more time we need. Often the first approach fails. Sometimes the first dozen approaches fail!

Here is another question, the famous "Census-Taker Problem." A few people might think of this as an exercise, but for most, it is a problem.

Example 1.1.3 A census-taker knocks on a door, and asks the woman inside how many children she has and how old they are.

"I have three daughters, their ages are whole numbers, and the product of the ages is 36," says the mother.

"That's not enough information," responds the census-taker.

"I'd tell you the sum of their ages, but you'd still be stumped."

"I wish you'd tell me something more."

"Okay, my oldest daughter Annie likes dogs."

What are the ages of the three daughters?

After the first reading, it seems impossible—there isn't enough information to determine the ages. That's why it is a problem, and a fun one, at that. (The answer is at the end of this chapter, on page 12, if you get stumped.)

If the Census-Taker Problem is too easy, try this next one (see page 75 for solution):

Example 1.1.4 I invite 10 couples to a party at my house. I ask everyone present, including my wife, how many people they shook hands with. It turns out that everyone questioned—I didn't question myself, of course—shook hands with a different number of people. If we assume that no one shook hands with his or her partner, how many people did my wife shake hands with? (I did not ask myself any questions.)

A good problem is mysterious and interesting. It is mysterious, because at first you don't know how to solve it. If it is not interesting, you won't think about it much. If it is interesting, though, you will want to put a lot of time and effort into understanding it.

This book will help you to investigate and solve problems. If you are an inexperienced problem solver, you may often give up quickly. This happens for several reasons.

- You may just not know how to begin.

- You may make some initial progress, but then cannot proceed further.
- You try a few things, nothing works, so you give up.

An experienced problem solver, in contrast, is rarely at a loss for how to begin investigating a problem. He or she[1] confidently tries a number of approaches to get started. This may not solve the problem, but some progress is made. Then more specific techniques come into play. Eventually, at least some of the time, the problem is resolved. The experienced problem solver operates on three different levels:

Strategy: Mathematical and psychological ideas for starting and pursuing problems.

Tactics: Diverse mathematical methods that work in many different settings.

Tools: Narrowly focused techniques and "tricks" for specific situations.

1.2 The Three Levels of Problem Solving

Some branches of mathematics have very long histories, with many standard symbols and words. Problem solving is not one of them.[2] We use the terms **strategy**, **tactics** and **tools** to denote three different levels of problem solving. Since these are not standard definitions, it is important that we understand exactly what they mean.

A Mountaineering Analogy

You are standing at the base of a mountain, hoping to climb to the summit. Your first *strategy* may be to take several small trips to various easier peaks nearby, so as to observe the target mountain from different angles. After this, you may consider a somewhat more focused strategy, perhaps to try climbing the mountain via a particular ridge. Now the *tactical* considerations begin: how to actually achieve the chosen strategy. For example, suppose that strategy suggests climbing the south ridge of the peak, but there are snowfields and rivers in our path. Different tactics are needed to negotiate each of these obstacles. For the snowfield, our tactic may be to travel early in the morning, while the snow is hard. For the river, our tactic may be scouting the banks for the safest crossing. Finally, we move onto the most tightly focused level, that of *tools*: specific techniques to accomplish specialized tasks. For example, to cross the snowfield we may set up a particular system of ropes for safety and walk with ice axes. The river crossing may require the party to strip from the waist down and hold hands for balance. These are all tools. They are very specific. You would never summarize, "To climb the mountain we had to take our pants off and hold hands," because this was a minor—though essential—component of the entire climb. On the other hand, strategic and sometimes tactical ideas are often described in your summary: "We decided to reach the summit via the south ridge and had to cross a difficult snowfield and a dangerous river to get to the ridge."

[1] We will henceforth avoid the awkward "he or she" construction by alternating genders in subsequent chapters.

[2] In fact, there does not even exist a standard name for the theory of problem solving, although George Pólya and others have tried to popularize the term **heuristics** (see, for example, [32]).

As we climb a mountain, we may encounter obstacles. Some of these obstacles are easy to negotiate, for they are mere exercises (of course this depends on the climber's ability and experience). But one obstacle may present a difficult miniature problem, whose solution clears the way for the entire climb. For example, the path to the summit may be easy walking, except for one 10-foot section of steep ice. Climbers call negotiating the key obstacle the **crux move**. We shall use this term for mathematical problems as well. A crux move may take place at the strategic, tactical or tool level; some problems have several crux moves; many have none.

From Mountaineering to Mathematics

Let's approach mathematical problems with these mountaineering ideas. When confronted with a problem, you cannot immediately solve it, for otherwise, it is not a problem but a mere exercise. You must begin a process of **investigation**. This investigation can take many forms. One method, by no means a terrible one, is to just randomly try whatever comes into your head. If you have a fertile imagination, and a good store of methods, and a lot of time to spare, you may eventually solve the problem. However, if you are a beginner, it is best to cultivate a more organized approach. First, think strategically. Don't try immediately to solve the problem, but instead think about it on a less focused level. The goal of strategic thinking is to come up with a plan that may only barely have mathematical content, but which leads to an "improved" situation, not unlike the mountaineer's strategy, "If we get to the south ridge, it looks like we will be able to get to the summit."

Strategies help us get started, and help us continue. But they are just vague outlines of the actual work that needs to be done. The concrete tasks to accomplish our strategic plans are done at the lower levels of tactic and tool.

Here is an example that shows the three levels in action, from a 1926 Hungarian contest.

Example 1.2.1 Prove that the product of four consecutive natural numbers cannot be the square of an integer.

Solution: Our initial strategy is to familiarize ourselves with the statement of the problem, i.e., to **get oriented**. We first note that the question asks us to prove something. Problems are usually of two types—those that ask you to prove something and those that ask you to find something. The Census-Taker problem (Example 1.1.3) is an example of the latter type.

Next, observe that the problem is asking us to prove that something *cannot* happen. We divide the problem into **hypothesis** (also called "the given") and **conclusion** (whatever the problem is asking you to find or prove). The hypothesis is:

Let n be a natural number.

The conclusion is:

$n(n+1)(n+2)(n+3)$ *cannot be the square of an integer.*

Formulating the hypothesis and conclusion isn't a triviality, since many problems don't state them precisely. In this case, we had to introduce some notation. Sometimes our choice of notation can be critical.

Perhaps we should focus on the conclusion: how do you go about showing that something cannot be a square? This strategy, trying to think about what would immediately lead to the conclusion of our problem, is called looking at the **penultimate step**.[3] Unfortunately, our imagination fails us—we cannot think of any easy criteria for determining when a number cannot be a square. So we try another strategy, one of the best for beginning just about any problem: **get your hands dirty**. We try plugging in some numbers to experiment with. If we are lucky, we may see a pattern. Let's try a few different values for n. Here's a table. We use the abbreviation $f(n) = n(n+1)(n+2)(n+3)$.

n	1	2	3	4	5	10
$f(n)$	24	120	360	840	1680	17160

Notice anything? The problem involves squares, so we are sensitized to look for squares. Just about everyone notices that the first two values of $f(n)$ are one less than a perfect square. A quick check verifies that additionally,

$$f(3) = 19^2 - 1, \quad f(4) = 29^2 - 1, \quad f(5) = 41^2 - 1, \quad f(10) = 131^2 - 1.$$

We confidently conjecture that $f(n)$ is one less than a perfect square for every n. Proving this conjecture is the penultimate step that we were looking for, *because a positive integer that is one less than a perfect square cannot be a perfect square* since the sequence $1, 4, 9, 16, \ldots$ of perfect squares contains no consecutive integers (the gaps between successive squares get bigger and bigger). Our new strategy is to prove the conjecture.

To do so, we need help at the tactical/tool level. We wish to prove that for each n, the product $n(n+1)(n+2)(n+3)$ is one less than a perfect square. In other words, $n(n+1)(n+2)(n+3)+1$ must be a perfect square. How to show that an algebraic expression is always equal to a perfect square? One tactic: **factor** the expression! We need to manipulate the expression, always keeping in mind our goal of getting a square. So we focus on putting parts together that are almost the same. Notice that the product of n and $n+3$ is "almost" the same as the product of $n+1$ and $n+2$, in that their first two terms are both $n^2 + 3n$. After regrouping, we have

$$[n(n+3)][(n+1)(n+2)] + 1 = (n^2+3n)(n^2+3n+2) + 1. \tag{1}$$

Rather than multiply out the two almost-identical terms, we introduce a little **symmetry** to bring squares into focus:

$$(n^2+3n)(n^2+3n+2)+1 = \Big((n^2+3n+1)-1\Big)\Big((n^2+3n+1)+1\Big)+1.$$

Now we use the "difference of two squares" factorization (a tool!) and we have

$$\begin{aligned}\Big((n^2+3n+1)-1\Big)\Big((n^2+3n+1)+1\Big)+1 &= (n^2+3n+1)^2 - 1 + 1\\ &= (n^2+3n+1)^2.\end{aligned}$$

[3]The word "penultimate" means "next to last."

We have shown that $f(n)$ is one less than a perfect square for all integers n, namely

$$f(n) = (n^2 + 3n + 1)^2 - 1,$$

and we are done. ■

Let us look back and analyze this problem in terms of the three levels. Our first strategy was orientation, reading the problem carefully and classifying it in a preliminary way. Then we decided on a strategy to look at the penultimate step that did not work at first, but the strategy of numerical experimentation led to a conjecture. Successfully proving this involved the tactic of factoring, coupled with a use of symmetry and the tool of recognizing a common factorization.

The most important level was strategic. Getting to the conjecture was the crux move. At this point the problem metamorphosed into an exercise! For even if you did not have a good tactical grasp, you could have muddled through. One fine method is **substitution**: Let $u = n^2 + 3n$ in equation (1). Then the right-hand side becomes $u(u+2) + 1 = u^2 + 2u + 1 = (u+1)^2$. Another method is to multiply out (ugh!). We have

$$n(n+1)(n+2)(n+3) + 1 = n^4 + 6n^3 + 11n^2 + 6n + 1.$$

If this is going to be the square of something, it will be the square of the quadratic polynomial $n^2 + an + 1$ or $n^2 + an - 1$. Trying the first case, we equate

$$n^4 + 6n^3 + 11n^2 + 6n + 1 = (n^2 + an + 1)^2 = n^4 + 2an^3 + (a^2 + 2)n^2 + 2an + 1$$

and we see that $a = 3$ works; i.e., $n(n+1)(n+2)(n+3) + 1 = (n^2 + 3n + 1)^2$. This was a bit less elegant than the first way we solved the problem, but it is a fine method. Indeed, it teaches us a useful *tool*: the method of **undetermined coefficients**.

1.3 A Problem Sampler

The problems in this book are classified into three large families: **recreational**, **contest** and **open-ended**. Within each family, problems split into two basic kinds: problems **"to find"** and problems **"to prove."**[4] Problems "to find" ask for a specific piece of information, while problems "to prove" require a more general argument. Sometimes the distinction is blurry. For example, Example 1.1.4 above is a problem "to find," but its solution may involve a very general argument.

What follows is a descriptive sampler of each family.

Recreational Problems

Also known as "brain teasers," these problems usually involve little formal mathematics, but instead rely on creative use of basic strategic principles. They are excellent to work on, because no special knowledge is needed, and any time spent thinking about a

[4]These two terms are due to George Pólya [32].