### Chapter 5

**Norms**

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Introduction

This chapter discusses vector norms and matrix norms in more generality and applies them to the Procrustes problem.

The reading for this chapter is [1, §7.2-7.4].

The Tikhonov regularized least-squares problem in Ch. 4 illustrates the two primary uses of norms:

\[
\hat{x} = \arg \min_x \|Ax - y\|_2^2 + \beta \|x\|_2^2.
\]

distance: how far size: how big

5.0 Vector norms

So far the only vector norm discussed in these notes has been the common Euclidean norm. Many other vector norms are also important in signal processing.

Define. [1, p. 57] A norm on a vector space \( \mathcal{V} \) defined over a field \( \mathbb{F} \) is a function \( \| \cdot \| \) from \( \mathcal{V} \) to \([0, \infty)\) that satisfies the following properties \( \forall x, y \in \mathcal{V} : \)

- \( \|x\| \geq 0 \) (nonnegative)
- \( \|x\| = 0 \) iff \( x = 0 \) (positive)
- \( \|\alpha x\| = |\alpha| \|x\| \) for all scalars \( \alpha \in \mathbb{F} \) in the field (homogeneous)
- \( \|x + y\| \leq \|x\| + \|y\| \) (triangle inequality)
Examples of vector norms

- For $1 \leq p < \infty$, the $\ell_p$ norm is

  $$\|x\|_p \triangleq \left( \sum_i |x_i|^p \right)^{1/p}.$$  \hspace{1cm} (5.1)

  - The vector 2-norm or Euclidian norm is the case $p = 2$: $\|x\|_2 \triangleq \sqrt{\sum_i |x_i|^2}$.
  - The 1-norm or “Manhattan norm” is the case $p = 1$: $\|x\|_1 \triangleq \sum_i |x_i|$.

- The max norm or infinity norm or $\ell_\infty$ norm is

  $$\|x\|_\infty \triangleq \sup \{|x_1|, |x_2|, \ldots\},$$  \hspace{1cm} (5.2)

  where sup denotes the supremum (least upper bound) of a set. One can show [2, Prob. 2.12] that

  $$\|x\|_\infty = \lim_{p \to \infty} \|x\|_p.$$  \hspace{1cm} (5.3)

  For the vector space $\mathbb{F}^N$, the supremum is simply a maximum:

  $$\|x\|_\infty \triangleq \max \{|x_1|, \ldots, |x_N|\},$$  \hspace{1cm} (5.4)
• For quantifying sparsity, it is useful to note that

$$\lim_{p \to 0} \|x\|_p^p = \sum_i I\{x_i \neq 0\} \triangleq \|x\|_0,$$

where $I\{\cdot\}$ denotes the indicator function that is unity if the argument is true and zero if false. However, the “0-norm” $\|x\|_0$ is not a vector norm because it does not satisfy the at least one of the conditions of the norm definition above. The proper name for $\|x\|_0$ is counting measure.

Which of the four properties of a vector norm does $\|\cdot\|_0$ satisfy?
A: 1,2 
B: 1,3 
C: 1,2,3 
D: 1,2,4 
E: 1,3,4

Practical implementation

For the preceding examples, in JULIA use:

```
vecnorm(v, p)  vecnorm(v, 2)  vecnorm(v, 1)
vecnorm(v, Inf) vecnorm(v, 0)
```

Caution. Using `norm(v, Inf)` and `norm(v', Inf)` yield different results in general! (See matrix norms below.) It is more clear to use `vecnorm()`.

Caution. For $p < 1$, $\|\cdot\|_p$ is not a norm, though it is sometimes used in practical problems and `vecnorm` will evaluate (5.1) for any $-\infty \leq p \leq \infty$. 


Properties of norms

- Let $\| \cdot \|_\alpha$ and $\| \cdot \|_\beta$ be any two vector norms on a finite-dimensional space. Then there exist finite positive constants $C_m$ and $C_M$ (that depend on $\alpha$ and $\beta$) such that:

$$C_m \| \cdot \|_\alpha \leq \| \cdot \|_\beta \leq C_M \| \cdot \|_\alpha. \quad (5.6)$$

In a sense then “all norms are equivalent” to within constant factors.

- For any vector norm, the reverse triangle inequality is:

$$| \| x \| - \| y \| | \leq \| x - y \|, \quad \forall x, y \in \mathcal{V}.$$

- All vector norms are convex functions:

$$\| \alpha x + (1 - \alpha)z \| \leq \alpha \| x \| + (1 - \alpha) \| z \|, \quad \forall \alpha \in [0, 1], \quad \forall x, z \in \mathcal{V}. $$

This fact is easy to prove using the triangle inequality and the homogeneity property (HW).

- For $p > 1$, $f(x) \triangleq \| x \|_p$ is strictly convex.

Norm notation

Some math literature uses $|x|$ instead of $\| x \|$ to denote a vector norm. That notation should be avoided for matrices where $|A|$ often denotes the determinant of $A$. Sometimes one must determine from context what $| \cdot |$ means in such literature.
Unitarily invariant norms

Some vector norms have the following useful property.

Define. A vector norm $\|\cdot\|$ on $\mathbb{F}^N$ is **unitarily invariant** iff for every unitary matrix $U \in \mathbb{F}^{N \times N}$:

$$\|UX\| = \|X\|, \quad \forall X \in \mathbb{F}^N.$$ 

Example. The Euclidean norm $\|\cdot\|_2$ on $\mathbb{F}^N$ is unitarily invariant, because for any unitary $U$ (see p. 1.42):

$$\|UX\|_2 = \sqrt{(UX)'(UX)} = \sqrt{x'U'UX} = \sqrt{x'x} = \|X\|_2, \quad \forall X \in \mathbb{F}^N.$$ 

As noted previously, this property is related to **Parseval’s theorem**.

Example. $\|\cdot\|_1$ is not unitarily invariant.

If $U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $x = e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ then $\|x\|_1 = 1$ but $\|UX\|_1 = \sqrt{2}$. 

Inner products

Most of the vector spaces used in this course are inner product spaces, meaning a vector space with an associated inner product operation.

Define. For a vector space $V$ over the field $F$, an inner product operation is a function $\langle \cdot, \cdot \rangle : V \times V \rightarrow F$ that must satisfy the following axioms $\forall x, y \in V, \alpha \in F$.

- $\langle x, y \rangle = \langle y, x \rangle^*$ (Hermitian symmetry)
- $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ (additivity)
- $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$ (scaling)
- $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ iff $x = 0$. (positive definite)

Examples of inner products

Example. For vectors in $F^N$, the usual inner product is

$$\langle x, y \rangle = \sum_{n=1}^{N} x_n y_n^*.$$  

Example. For the (infinite dimensional) vector space of square integrable functions on the interval $[a, b]$, the following integral is a valid inner product:

$$\langle f, g \rangle = \int_{a}^{b} f(t)g^*(t) \, dt.$$
Example. For two matrices $A, B \in \mathbb{F}^{M \times N}$ (a “vector space!”), the Frobenius inner product is:

$$\langle A, B \rangle = \text{trace}\{AB'\}.$$

Exercise. Verify the four properties above for these inner product examples.

**Properties of inner products**

- Bilinearity:

$$\langle \sum_i \alpha_i x_i, \sum_j \beta_j y_j \rangle = \sum_i \sum_j \alpha_i \beta_j^* \langle x_i, y_j \rangle, \quad \forall \{x_i\}, \{y_j\} \in V.$$

- Any valid vector inner product induces a valid vector norm:

$$\|x\| = \sqrt{\langle x, x \rangle}. \quad (5.7)$$

- A vector norm satisfies the parallelogram law:

$$\frac{1}{2} (\|x + y\|^2 + \|x - y\|^2) = \|x\|^2 + \|y\|^2, \quad \forall x, y \in V,$$

iff it is induced by an inner product via (5.7). The required inner product is:

$$\langle x, y \rangle \triangleq \frac{1}{4} \left( \|x + y\|^2 - \|x - y\|^2 + i \|x + iy\|^2 - i \|x - iy\|^2 \right) = \frac{\|x + y\|^2 - \|x\|^2 - \|y\|^2}{2} + i \frac{\|x + iy\|^2 - \|x\|^2 - \|y\|^2}{2}.$$
• The **Cauchy-Schwarz inequality** (or **Schwarz** or **Cauchy-Bunyakovsky-Schwarz inequality**) states:

\[
|\langle x, y \rangle| \leq \|x\| \|y\| = \sqrt{\langle x, x \rangle} \sqrt{\langle y, y \rangle}, \quad \forall x, y \in V,
\]

for a norm \(\|\cdot\|\) induced by an inner product \(\langle \cdot, \cdot \rangle\) via (5.7), with equality iff \(x\) and \(y\) are linearly dependent.

**Proof of Cauchy-Schwarz inequality for** \(\mathbb{F}^N\)

For any \(x, y \in \mathbb{F}^N\) let \(A = [x \ y]\), so \(A^T A = \begin{bmatrix} x'x & x'y \\ y'x & y'y \end{bmatrix} \).

\(A^T A\) is Hermitian symmetric \(\implies\) its eigenvalues are all real and nonnegative.

\(\implies \det\{A^T A\} \geq 0 \implies (x'x)(y'y) - (y'x)(x'y) \geq 0 \implies |x'y|^2 \leq (x'x)(y'y) = \|x\|^2 \|y\|^2.\)

Taking the square root of both sides yields the inequality.

We used the fact that \((y'x)(x'y) = \langle x, y \rangle \langle y, x \rangle = \langle x, y \rangle (\langle x, y \rangle)^* = |\langle x, y \rangle|^2.\)
Angle between vectors

Define. The angle $\theta$ between two nonzero vectors $x, y \in V$ is defined by

$$\cos \theta = \frac{|\langle x, y \rangle|}{\|x\| \|y\|} \Rightarrow \theta \in [0, \pi/2].$$

The Cauchy-Schwarz inequality is equivalent to the statement $|\cos \theta| \leq 1$.

What is the angle between two orthogonal vectors?
A: 0  B: 1  C: $\pi/2$  D: $\pi$  ??

An inner product for random variables

For two real, zero-mean random variables $X, Y$ defined on a joint probability space, a natural inner product is $E[XY]$. (Keep in mind that random variables are functions.) With this definition, the corresponding norm is $\|X\| = \sqrt{\langle X, X \rangle} = \sqrt{E[X^2]} = \sigma_X$, the standard deviation of $X$. Here, the Cauchy-Schwarz inequality is equivalent to usual bound on the correlation coefficient: $\rho_{X,Y} \triangleq \frac{E[XY]}{\sigma_X \sigma_Y} \Rightarrow |\rho_{X,Y}| \leq 1$.

With this definition of inner product, what types of random variables are “orthogonal”? Pairs of random variables that are uncorrelated, i.e., where $E[XY] = 0$. 
Also important are **matrix norms**; these quantify “how large” are the elements of a matrix.

Define. [1, p. 59] A norm on the vector space of matrices $\mathbb{F}^{M \times N}$ is a function $\| \cdot \|$ from $\mathbb{F}^{M \times N}$ to $[0, \infty)$ that satisfies the following properties $\forall A, B \in \mathbb{F}^{M \times N}$:

- $\|A\| \geq 0$ (nonnegative)
- $\|A\| = 0$ iff $A = 0_{M \times N}$ (positive)
- $\|\alpha A\| = |\alpha| \|A\|$ for all scalars $\alpha \in \mathbb{F}$ in the field (homogeneous)
- $\|A + B\| \leq \|A\| + \|B\|$ (triangle inequality)

Because the set of all $M \times N$ matrices $\mathbb{F}^{M \times N}$ is itself a vector space, matrix norms are simply vector norms for that space. So at first having a new definition might seem to have modest utility. However, many, but not all, matrix norms are sub-multiplicative, also called **consistent** [1, p. 61], meaning that they satisfy the following inequality:

$$\|AB\| \leq \|A\| \|B\|, \quad \forall A, B.$$  \hfill (5.9)

These notes use the notation $\| \cdot \|$ to distinguish such **matrix norms** from the ordinary matrix norms $\| \cdot \|$ on the vector space $\mathbb{F}^{M \times N}$ that need not satisfy this extra condition.
Examples of matrix norms

- The **max norm** on $\mathbb{F}^{M\times N}$ is the element-wise maximum: $\|A\|_{\max} \triangleq \max_{i,j} |a_{ij}|$. This norm is somewhat like the infinity norm for vectors. One can compute it in JULIA using `vecnorm(A, Inf)`. However, this differs completely from `norm(A, Inf)` that computes $\|A\|_{\infty}$ described below. The max norm is a matrix norm on the vector space $\mathbb{F}^{M\times N}$ but it does not satisfy the sub-multiplicative condition (5.9) so it is of limited use. Most of the norms of interest in signal processing are sub-multiplicative, so such matrix norms are our primary focus hereafter.

- The **Frobenius norm** (aka Hilbert-Schmidt norm and trace norm) is defined on $\mathbb{F}^{M\times N}$ by

$$
\|A\|_F \triangleq \sqrt{\sum_{m=1}^{M} \sum_{n=1}^{N} |a_{mn}|^2} = \sqrt{\text{trace}\{A^\prime A\}} = \sqrt{\text{trace}\{AA^\prime\}},
$$

(5.10)

and is also called the **Schur norm** and **Schatten 2-norm**. It is a very easy norm to compute. The equalities related to trace are a HW problem.
To relate the Frobenius norm of a matrix to its singular values:

\[
\|A\|_F = \sqrt{\text{trace}\{AA^\prime\}} = \sqrt{\text{trace}\{U_r \Sigma_r V_r^\prime V_r \Sigma_r U_r^\prime\}} = \sqrt{\text{trace}\{\Sigma_r \Sigma_r U_r^\prime U_r\}} = \sqrt{\text{trace}\{\Sigma^2_r\}} = \sqrt{\sum_{k=1}^{r} \sigma^2_k}.
\]

This norm is invariant to unitary transformations [3, p. 442], because of the trace property (1.13). This norm is induced by the **Frobenius inner product**. It is not induced by any vector norm on \(\mathbb{F}^N\) (see next page) [4], but nevertheless it is **compatible** with the Euclidean vector norm because

\[
\|Ax\|_2 \leq \|A\|_F \|x\|_2.
\]  

(5.11)

However, this upper bound is not tight in general. (It is tight for rank-1 matrices only.) By combining (5.11) with the definition of matrix multiplication, one can show easily that the Frobenius norm is **sub-multiplicative** [5, p. 291].

**What is \(\|uv\|_F\)?**

A: \(|u^\prime v|\)  
B: \(|u^\prime v|^2\)  
C: \(\sqrt{\|u\|_2 \|v\|_2}\)  
D: \(\|u\|_2 \|v\|_2\)  
E: None of these.
5.14 **Induced matrix norms**

If $\| \cdot \|$ is any vector norm that is suitable for both $\mathbb{F}^N$ and $\mathbb{F}^M$, then a matrix norm for $\mathbb{F}^{M \times N}$ is:

$$
\| A \| \triangleq \max_{x : \|x\| = 1} \| Ax \| = \max_{x \neq 0} \frac{\| Ax \|}{\| x \|}.
$$

(5.12)

By construction:

$$
\| Ax \| \leq \| A \| \| x \|, \quad \forall x \in \mathbb{F}^N.
$$

(5.13)

We say such a matrix norm $\| \cdot \|$ is **induced** by the vector norm $\| \cdot \|$. Importantly, the **sub-multiplicative** property (5.9) holds whenever the number of columns of $A$ matches the number of rows of $B$.

**Example.** The most important matrix norms are induced by the vector norm $\| \cdot \|_p$, *i.e.*

$$
\| A \|_p = \max_{x \neq 0} \frac{\| Ax \|_p}{\| x \|_p}.
$$

(5.14)

- The **spectral norm** $\| \cdot \|_2$, often denoted simply $\| \cdot \|$, is defined on $\mathbb{F}^{M \times N}$ by (5.14) with $p = 2$. This is the matrix norm induced by the Euclidean vector norm. As shown on p. 2.20:

$$
\| A \|_2 = \max_{x \neq 0} \frac{\| Ax \|_2}{\| x \|_2} = \max \left\{ \sqrt{\lambda} : \lambda \in \text{eig}\{A^tA\} \right\} = \sigma_1(A).
$$
• The **maximum row sum matrix norm** is defined on $\mathbb{F}^{M \times N}$ by

$$
\| A \|_\infty \triangleq \max_{1 \leq i \leq M} \sum_{j=1}^{N} |a_{ij}|.
$$

(5.15)

It is induced by the $\ell_\infty$ vector norm. It differs from the max norm defined above!

• The **maximum column sum matrix norm** is defined on $\mathbb{F}^{M \times N}$ by

$$
\| A \|_1 \triangleq \max_{x \neq 0} \frac{\| A x \|_1}{\| x \|_1} = \max_{1 \leq j \leq N} \sum_{i=1}^{M} |a_{ij}|.
$$

(5.16)

It is induced by the $\ell_1$ vector norm.
• For $1 \leq p \leq \infty$, the **Schatten p-norm** of a $M \times N$ matrix having rank $r$ is defined in terms of its nonzero singular values:

$$\|A\|_{S,p} = \left(\sum_{k=1}^{r} \sigma_k^p\right)^{1/p}.$$ 

All of these **Schatten norms** are **sub-multiplicative**.

Important special cases:
- $\|A\|_{S,1} = \|A\|_\ast$ is also called the **nuclear norm** and sometimes the **trace norm** [1, p. 60].
- $\|A\|_{S,2} = \|A\|_F$
- $\|A\|_{S,\infty} = \sigma_1(A) = \|A\|_2$

• The **Ky-Fan $K$-norm** is the sum of the first $K$ singular values of a matrix:

$$\|A\|_{\text{Ky–Fan},K} = \sum_{k=1}^{K} \sigma_k(A).$$

Challenge. Determine if the Ky-Fan $K$-norm is sub-multiplicative, or find a counter-example. For a PCA generalization that uses this norm see [6].
Practical implementation

JULIA commands for some of these norms are as follows:

<table>
<thead>
<tr>
<th>Norm</th>
<th>JULIA Command</th>
</tr>
</thead>
<tbody>
<tr>
<td>$|A|_1$</td>
<td>norm(A, 1)</td>
</tr>
<tr>
<td>$|A|_2$</td>
<td>norm(A, 2)</td>
</tr>
<tr>
<td>$|A|_\infty$</td>
<td>norm(A, Inf)</td>
</tr>
<tr>
<td>$|A|_*$</td>
<td>sum(svd(A)[2]) or sum(svdfact(A)[:S])</td>
</tr>
</tbody>
</table>

(||A||_{\text{max}} is vecnorm(A, Inf))

Again, caution that $\|A\|_{\text{Inf}}$ differs from vecnorm(A, Inf)

Examples

For $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$, what is vecnorm(A, Inf) ?

A: 1  B: 2  C: 3  D: 6  E: None of these

For $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$, what is norm(A, Inf) ?

A: 1  B: 2  C: 3  D: 6  E: None of these

For $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$, what is norm(A, 2) ?

Hint: perhaps recall the eigenvalue commutative property (1.12).

A: 1  B: 2  C: 3  D: 6  E: None of these
Properties of matrix norms

All matrix norms are also equivalent (to within constants that depend on the matrix dimensions). See [1, p. 61] for inequalities relating various matrix norms.

Example. (See HW): \( \mathbf{A} \in \mathbb{R}^{M \times N} \implies \|\mathbf{A}\|_1 \leq \sqrt{N} \|\mathbf{A}\|_2 \).

Example. To relate the spectral norm and nuclear norm for a matrix \( \mathbf{A} \) having rank \( r \):

\[
\|\mathbf{A}\|_* = \sum_{k=1}^{r} \sigma_k \leq \sum_{k=1}^{r} \sigma_1 = r \sigma_1 = r \|\mathbf{A}\|_2 \\
\|\mathbf{A}\|_2 = \sigma_1 \leq \sum_{k=1}^{r} \sigma_k = \|\mathbf{A}\|_*
\]

Combining:

\[
\frac{1}{r} \|\mathbf{A}\|_* \leq \|\mathbf{A}\|_2 \leq \|\mathbf{A}\|_* \leq r \|\mathbf{A}\|_2
\]

To express it in way that depends on the norm only (not \( r \), which is a property of a specific matrix):

\[
\frac{1}{\min(M, N)} \|\mathbf{A}\|_* \leq \|\mathbf{A}\|_2 \leq \|\mathbf{A}\|_* \leq \min(M, N) \|\mathbf{A}\|_2
\]
Unitarily invariant matrix norms

Define. A matrix norm $\| \cdot \|$ on $\mathbb{F}^{M \times N}$ is called unitarily invariant iff for all unitary matrices $U \in \mathbb{F}^{M \times M}$ and $V \in \mathbb{F}^{N \times N}$:

$$\|UAV\| = \|A\|, \quad \forall A \in \mathbb{F}^{M \times N}.$$ 

Theorem.

- The spectral norm $\|A\|_2$ is unitarily invariant
- All Schatten p-norms $\|A\|_{S,p}$ are unitarily invariant
  
  Proof sketch: unitary matrix rotations do not change singular values.
- The Frobenius norm $\|A\|_F$ is unitarily invariant
  
  Proof for Frobenius case:

$$\|UAV\|_F = \sqrt{\text{trace}\{V'A'UVAV\}} = \sqrt{\text{trace}\{V'A'AV\}}$$
$$= \sqrt{\text{trace}\{VV'A'A\}} = \sqrt{\text{trace}\{AA'\}}$$
$$= \|A\|_F.$$ 

(We could also prove it using singular values.)
Spectral radius

Define. For any square matrix, the **spectral radius** is the maximum absolute eigenvalue:

\[ A \in \mathbb{F}^{N \times N} \implies \rho(A) \triangleq \max_i |\lambda_i(A)|. \]

- By construction, \(|\lambda_i(A)| \leq \rho(A)| so all eigenvalues lie within a disk in the complex plane of radius \(\rho(A)|, hence the name.
- In general, \(\rho(A)| is not a matrix norm and \(\|Ax\| \not\leq \rho(A) \|x\|.\)
- However, if \(A| is normal, then its eigenvalues are real, so assuming we order the eigenvalues in decreasing order of their absolute values, we can relate its orthogonal eigendecomposition to an SVD as follows:

\[ A = V \Lambda V' = \sum_{n=1}^{N} \lambda_n v_n v_n' = \sum_{n=1}^{N} |\lambda_n| \operatorname{sign}(\lambda_n) u_n v_n'. \]

- Thus \(A = A' \implies \rho(A) = \sigma_1(A) = \|A\|_2, so \(\|Ax\|_2 \leq \rho(A) \|x\|_2.\)
- Furthermore, \(A = A' \implies |x'Ax| \leq \rho(A) \|x\|_2^2| because

\[
\max_{x \neq 0} \frac{|x'Ax|}{\|x\|_2^2} = \max_{z \neq 0} \frac{|(Vz)'A(Vz)|}{\|Vz\|_2^2} = \max_{z \neq 0} \frac{|z'\Lambda z|}{\|z\|_2^2} = \max_n |\lambda_n(A)| = \rho(A) = \sigma_1(A) = \|A\|_2.
\]
• If $\| \cdot \|$ is any induced matrix norm on $\mathbb{F}^{N \times N}$ and if $A \in \mathbb{F}^{N \times N}$, then

$$\rho(A) \leq \| A \|. \tag{5.17}$$

Proof. If $Av = \lambda v$, then $|\lambda| \| v \| = \| \lambda v \| = \| Av \| \leq \| A \| \| v \|$. Dividing by $\| v \|$, which is fine because $v \neq 0$, yields $|\lambda| \leq \| A \|$. This inequality holds for all eigenvalues, including the one with maximum magnitude.

• For any $A \in \mathbb{F}^{N \times N}$, the spectral radius is an infimum of all induced matrix norms:

$$\rho(A) = \inf \{ \| A \| : \| \cdot \| \text{ is an induced matrix norm} \}.$$ \hfill \Box

• If $A \in \mathbb{F}^{N \times N}$, then $\lim_{k \to \infty} A^k = 0$ if and only if $\rho(A) < 1$.

This property is particularly important for analyzing the convergence of iterative algorithms, including training recurrent neural networks [7]. (cf. HW)

• **Gelfand’s formula** for any induced matrix norm $\| \cdot \|$ is:

$$\rho(A) = \lim_{k \to \infty} \| A^k \|^{1/k}. \tag{5.18}$$
Which equality (if any) correctly relates a singular value and a spectral radius for any general matrix $A \in \mathbb{F}^{M \times N}$?

A: $\sigma_1(A) \overset{?}{=} |\rho(A)|$

B: $\sigma_1(A) \overset{?}{=} \rho^2(A)$

C: $\sigma_1(A) \overset{?}{=} \rho(A'A)$

D: $\sigma_1(A) \overset{?}{=} \sqrt{\rho(A'A)}$

E: None of these.
5.2 Procrustes analysis

(An application of the SVD and the Frobenius matrix norm)

One use of matrix norms is quantifying the dissimilarity of two matrices by using the norm of their difference. We illustrate that use by solving the orthogonal Procrustes problem [8].

The goal of the Procrustes problem is to find an orthogonal matrix $Q$ in $\mathbb{R}^{M \times M}$ that makes two other matrices $B$ and $A$ in $\mathbb{R}^{M \times N}$ as similar as possible by “rotating” the columns of $A$:

$$
\hat{Q} = \arg \min_{Q : Q'Q = I_M} f(Q), \quad f(Q) \triangleq \| B - QA \|_F^2.
$$

(One could use some other norm but the Frobenius is simple and natural here.)
One of many motivating applications is performing image registration of two pictures of the same scene acquired with different camera orientations, using a technique called landmark registration.

Example. Here the goal is to match (by rotation) two sets of landmark coordinates:

\[
A = \begin{bmatrix}
-59 & -25 & 49 \\
6 & -33 & 20
\end{bmatrix}, \quad B = \begin{bmatrix}
-54.1 & -5.15 & 32.44 \\
-24.3 & -41.08 & 41.82
\end{bmatrix} \approx QA = \begin{bmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{bmatrix} A.
\]

Here \( M = 2 \) and \( N = 3 \) and typically \( M < N \) in such problems.
Analyze the cost function:

\[ f(Q) = \|B - QA\|_F^2 = \text{trace}\{(B - QA)'(B - QA)\} \]
\[ = \text{trace}\{B'B - A'Q'B - B'QA + A'Q'QA\} \]
\[ = \text{trace}\{B'B - A'Q'B - B'QA + A'A\} \]
\[ = \text{trace}\{B'B\} + \text{trace}\{A'A\} - \text{trace}\{A'Q'B\} - \text{trace}\{B'QA\} \]
\[ = \text{trace}\{B'B\} + \text{trace}\{A'A\} - 2\text{trace}\{A'Q'B\} \]
\[ = \text{trace}\{B'B\} + \text{trace}\{A'A\} - 2\text{trace}\{Q'BA'\} \]

So minimizing \( f(Q) \) is equivalent to maximizing

\[ g(Q) \triangleq \text{trace}\{Q'BA'\}. \]

Use an SVD (of course!) of the \( M \times M \) matrix \( C \triangleq BA' = U\Sigma V' \) so

\[ g(Q) = \text{trace}\{Q'BA'\} = \text{trace}\{Q'U\Sigma V'\} = \text{trace}\{V'Q'U\Sigma\} \]
\[ = \text{trace}\{W\Sigma\}, \quad W = W(Q) \triangleq V'Q'U. \]

Using the orthogonality of \( U, V \) and \( Q \), it is clear that the \( M \times M \) matrix \( W \) is orthogonal (cf. HW):

\[ W'W = U'QVV'Q'U = U'QI_M Q'U = U'U = I_M. \]
We must maximize $\text{trace}\{W\Sigma\}$ over $Q$ orthogonal, where $W$ depends on $Q$ but $\Sigma$ does not. Observe:

$$[W\Sigma]_{mm} = w_{mm}\sigma_m \implies \text{trace}\{W\Sigma\} = \sum_{m=1}^{M} w_{mm}\sigma_m.$$ 

To proceed, we look for an upper bound for this sum. Because $W$ is an orthogonal matrix, each of its columns have unit norm, i.e., $\sum m = 1^N |w_{mn}|^2 = 1$ for all $m$, so $w_{mn} \leq 1$ for all $m,n$. This inequality yields the following upper bound:

$$\text{trace}\{W\Sigma\} \leq \sum_{m=1}^{M} \sigma_m = \text{trace}\{I\Sigma\}.$$ 

This upper bound is achieved when $W = I$. Now solve for $Q$:

$$W = V'Q'U = I \implies VV'Q'UU' = VU' \implies Q' = VU' \implies \hat{Q} = UV'.$$

In summary:

$$\hat{Q} = \arg\min_{Q:Q'Q=I} \|B - QA\|_F^2 = UV', \text{ where } C = BA' = U\Sigma V'.$$  \hspace{1cm} (5.19)

A homework problem will express $C = QP$ where $P$ is positive semi-definite, using a polar decomposition or polar factorization [1, p. 41].
Sanity check (self consistency and scale invariance)

Suppose $B$ is exactly a rotated version of the columns of $A$, along with an additional scale factor \( \alpha \), i.e., $B = \alpha \tilde{Q}A$ for some orthogonal matrix $\tilde{Q}$; equivalently $A = \frac{1}{\alpha} \tilde{Q}'B$. We now verify that the Procrustes method finds the correct rotation, i.e., $\hat{Q} = \tilde{Q}$.

Let $B = \tilde{U}\tilde{\Sigma}\tilde{V}'$ denote the SVD of $B$. Then the SVD of $C$ is evident by inspection:

\[
C = BA' = \frac{1}{\alpha} BB'\tilde{Q} = \frac{1}{\alpha} \tilde{U}\tilde{\Sigma}\tilde{V}'\tilde{\Sigma}'\tilde{U}'\tilde{Q} = \tilde{U} \frac{1}{\alpha} \tilde{\Sigma} \tilde{V}'\tilde{\Sigma}'\tilde{U}'\tilde{Q}.
\]

The Procrustes solution is indeed correct (self consistent), and invariant to the scale parameter $\alpha$:

\[
\hat{Q} = UV' = (\tilde{U})(\tilde{U}'\tilde{Q}) = \tilde{Q}.
\]

After finding $\hat{Q}$, if we also want to estimate the scale, we can solve a least-squares problem:

\[
\arg \min_{\alpha} \| B - \alpha \hat{Q}A \|_F = \frac{\text{trace}\{ BA'\hat{Q}' \}}{\text{trace}\{A'A\}} = \frac{\text{trace}\{ U\Sigma V'VU' \}}{\text{trace}\{A'A\}} = \sum_{k=1}^{r} \frac{\sigma_k}{\|A\|_F^2},
\]

where \( \{\sigma_k\} \) are the singular values of $C = BA'$

A HW problem will explore a real-world image registration example.
Example. For determining 2D image rotation, even a single nonzero point in each image will suffice! For example, suppose the first point is at \((1, 0)\) and the second point is at \((x, y)\) where \(x = 5 \cos \phi\) and \(y = 5 \sin \phi\). (This example includes scaling by a factor of 5 just to illustrate the generality.) Then \(A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\), \(B = \begin{bmatrix} x \\ y \end{bmatrix}\) so

\[
C = BA' = \begin{bmatrix} 5 \cos \phi \\ 5 \sin \phi \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \end{bmatrix} = \begin{bmatrix} \cos \phi & -q_1 \sin \phi \\ \sin \phi & q_1 \cos \phi \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & q_2 \end{bmatrix} = \begin{bmatrix} \cos \phi & -q_1 \sin \phi \\ \sin \phi & q_1 \cos \phi \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & q_2 \end{bmatrix}, \quad q_1, q_2 \in \{\pm 1\}.
\]

Here \(C\) is a simple outer product so finding a (full!) SVD by hand was easy. In fact we found four SVDs, corresponding to different signs for \(u_2\) and \(v_2\). For each of these SVDs, the optimal rotation matrix per (5.19) is

\[
\hat{Q} = UV' = \begin{bmatrix} \cos \phi & -q_1 \sin \phi \\ \sin \phi & q_1 \cos \phi \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & q_2 \end{bmatrix} = \begin{bmatrix} \cos \phi & -q \sin \phi \\ \sin \phi & q \cos \phi \end{bmatrix},
\]

where \(q \triangleq q_1q_2 \in \{\pm 1\}\). The two Procrustes solutions here (for \(q = \pm 1\)) both have the correct \(\cos \phi\) in the upper left and both exactly satisfy \(B = 5QA\).

So there are two Procrustes solutions that fit the data exactly, one of which (for \(q = 1\)) corresponds to a rotation matrix, and the other of which (for \(q = -1\)) has sign flip for the second coordinate.

In 2D, any rotation matrix is a unitary matrix, but the converse is not true!

Exercise. Explore what happens with two points: colinear, symmetric around zero, non-colinear.
Complex case - with translation

In many practical applications of the Procrustes problem, there can be both rotation and an unknown translation between the two sets of coordinates. Instead of the model \( B \sim Q A \), a more realistic model is \( B \sim Q A + d \) where \( d \in \mathbb{F}^M \) is an unknown displacement vector. In matrix form:

\[
B \approx QA + d1'_N.
\]

Now we must determine both an orthogonal matrix \( Q \in \mathbb{F}^{M \times M} \) and the vector \( d \in \mathbb{C}^M \) by a double minimization using a Frobenius norm:

\[
(\hat{Q}, \hat{d}) \triangleq \arg\min_{Q: Q'Q=I_M} \arg\min_{d \in \mathbb{F}^M} g(d, Q), \quad g(d, Q) \triangleq \| B - (QA + d1'_N) \|_{F}^2.
\]

We first focus on the inner minimization over the displacement \( d \) for any given \( Q \):

\[
g(d, Q) = \| B - (QA + d1'_N) \|_{F}^2 = \text{trace}\{(Z - d1'_N)'(ZA - d1'_N)\}, \quad Z \triangleq B - QA
\]

\[
= \text{trace}\{Z'Z\} - \text{trace}\{Z'd1'_N\} - \text{trace}\{1_Nd'Z\} + \text{trace}\{1_Nd'd1'_N\}
\]

\[
= \text{trace}\{Z'Z\} - \text{trace}\{1'_NZ'd\} - \text{trace}\{d'Z1_N\} + \text{trace}\{d'd1'_N1_N\}
\]

\[
= \text{trace}\{Z'Z\} - 1'_NZ'd - d'Z1_N + Nd'd
\]

\[
= \text{trace}\{Z'Z\} - \frac{1}{N} \| Z1_N \|_2^2 + N \| d - \frac{1}{N} Z1_N \|_2^2.
\]
It is clear from this expression that the optimal estimate of the displacement $d$ for any $Q$ is:

$$\hat{d}(Q) = \frac{1}{N}Z1_N = \frac{1}{N}(B - QA)1_N.$$ 

Now to find the optimal rotation matrix $Q$ we must solve the outer minimization:

$$\hat{Q} \triangleq \arg \min_{Q:Q'Q=I_M} g\left(\hat{d}(Q), Q\right)$$

$$g\left(\hat{d}(Q), Q\right) = \text{trace}\{(B - QA)'(B - AQ)\} - \frac{1}{N} \|B - QA\|_2^2$$

$$= -2 \text{real}\{\text{trace}\{Q'BA'\}\} + \frac{2}{N} \text{real}\{1_N' A'Q'B 1_N\}$$

$$= -2 \text{real}\{\text{trace}\{Q'BA'\}\} + 2 \text{real}\{\text{trace}\{Q'B1_N 1_N'A'\}\}$$

$$= -2 \text{real}\{\text{trace}\{Q'BMA'\}\}, \quad M \triangleq I - \frac{1}{N}1_N'1_N.$$ 

Following the previous logic, the optimal $Q$ is

$$Q = UV', \quad \text{where } \hat{C} \triangleq BMA' = U\Sigma V'.$$

The matrix $M$ is called a “de-meaning” or “centering” operator because $y = Mx$ subtracts the mean of $x$ from each element of $x$. In code: $y = x - \text{mean}(x)$
The de-meaning matrix $M$ is a symmetric idempotent matrix so $M = MM'$ and we can rewrite $\tilde{C}$ above as $\tilde{C} = (BM)(AM)' = \tilde{B} \tilde{A}'$ where $\tilde{A} \triangleq AM$, $\tilde{B} \triangleq BM$ are versions of $A$ and $B$ where each column has its mean subtracted out.

In words, to find the optimal rotation matrix when there is possible translation, we first de-mean each column of $A$ and $B$, and then compute the usual SVD of $\tilde{B} \tilde{A}'$ and use the left and right bases via $Q = UV'$.

**Exercise.** Do a sanity check in the case where $B = \alpha QA + d1_N'$.

**What is $M(\alpha 1_N)$?**

A: 0  B: 1  C: $I$  D: None of these.

---

**Practical implementation**

The solution to the Procrustes problem requires just a couple Julia statements. The key ingredient is simply the `svd` command.
5.3 Convergence of sequences of vectors and matrices

In later chapters we will be discussing iterative optimization algorithms and analyzing when such algorithms converge.

Convergence of a sequence of numbers

Define. We say a sequence of (possibly complex) numbers \( \{x_k\} \) converges to a limit \( x^* \) iff \( |x_k - x^*| \to 0 \) as \( k \to \infty \), where \( |\cdot| \) denotes absolute value (or complex magnitude more generally). Specifically,

\[
\forall \epsilon > 0, \quad \exists N_\epsilon \in \mathbb{N} \text{ s.t. } |x_k - x^*| < \epsilon \quad \forall k \geq N_\epsilon
\]

We now define convergence of a sequence of vectors or matrices by using a norm to quantify distance, relating to convergence of a sequence of scalars.

Define. We say a sequence of vectors \( \{x_k\} \) in a vector space \( \mathcal{V} \) converges to a limit \( x^* \in \mathcal{V} \) iff \( \|x_k - x^*\| \to 0 \) for some norm \( \|\cdot\| \) as \( k \to \infty \). Specifically,

\[
\forall \epsilon > 0, \quad \exists N_\epsilon \in \mathbb{N} \text{ s.t. } \|x_k - x^*\| < \epsilon \quad \forall k \geq N_\epsilon
\]

A matrix is simply a point in a vector space of matrices so we use essentially the same definition of convergence of a sequence of matrices:
Define. We say a sequence of matrices \( \{ \mathbf{X}_k \} \) (in a vector space \( \mathcal{V} \) of matrices) **converges** to a limit \( \mathbf{X}_* \in \mathcal{V} \) iff \( \| \mathbf{X}_k - \mathbf{X}_* \| \to 0 \) for some (matrix) norm \( \| \cdot \| \) as \( k \to \infty \). Specifically,

\[
\forall \epsilon > 0, \quad \exists N_\epsilon \in \mathbb{N} \text{ s.t. } \| \mathbf{X}_k - \mathbf{X}_* \| < \epsilon \quad \forall k \geq N_\epsilon
\]

**Example.** Consider (for simplicity) the sequence of **diagonal** matrices \( \{ \mathbf{D}_k \} \) defined by

\[
\mathbf{D}_k = \begin{bmatrix}
3 + 2^{-k} & 0 \\
0 & (-1)^k / k^2
\end{bmatrix}.
\]

This sequence converges to the limit \( \mathbf{D}_* = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} \) because

\[
\| \mathbf{D}_k - \mathbf{D}_* \|_F = \left\| \begin{bmatrix} 2^{-k} & 0 \\ 0 & (-1)^k / k^2 \end{bmatrix} \right\|_F = \sqrt{4^{-k} + 1/k^4} \to 0.
\]
The Procrustes problem has an SVD-based solution:

\[ \hat{Q} = \arg \min_{Q: Q'Q = I_M} \| B - QA \|_F^2 = UV', \quad C = BA' = U\Sigma V'. \]

- The solution is invariant to scaling factors \( \alpha QA \)
- Unknown displacement (translation) simply requires de-meaning \( A \) and \( B \) before doing SVD
- Displacement estimate (if needed) is \( \frac{1}{N} \left( B - \hat{Q}A \right) 1_N \).

Deriving the solution to this problem used many of the tools discussed so far: Frobenius norm, matrix trace and its properties, SVD, matrix/vector algebra.

**Exercise.** Suppose \( A \) and \( B \) are both real \( 1 \times N \) vectors (each with mean 0 for simplicity). What does Procrustes solution mean in this case geometrically?

*Hint:* What is SVD of \( BA' \) here?

If \( A = x' \) and \( B = y' \) where \( x, y \in \mathbb{R}^N \), then \( BA' = y'x = \text{sgn}(y'x)|y'x| \frac{1}{\sigma_1} \\\n\Rightarrow Q = UV' = \text{sgn}(y'x) = \pm 1 \). Here the “rotation” is just possibly negating the sign to match in 1D.

**Exercise.** What if \( B = e^{i\phi} A \)?

( in class if possible)
Challenge (for much later in the course). The Frobenius norm is not robust to outliers. Using something like an $\ell_1$ norm instead would provide better robustness [9].

**Bibliography**


