Problem 1

In Homework 8 we focused on solving sparse regression problems by incorporating an $\ell_1$-norm regularizer. If we want to encourage a different type of behavior, we can consider different regularizers. Moreover, there are structures in signals besides sparsity that still allow recovery in the case of highly under-determined systems. One popular regularizer is called the total variation (TV), which has applications in image denoising and inpainting. For a signal $w \in \mathbb{R}^D$, the TV is defined as

$$TV(w) = \sum_{i=1}^{D-1} |w_{i+1} - w_i|,$$

where $w_i$ denotes the $i$th element of $w$. In this problem, you will implement TV-regularized regression, which is useful for estimating signals that are known to be piecewise constant, as shown in Fig. 1 below.

(a) Define a matrix $C \in \mathbb{R}^{D-1 \times D}$ such that

$$TV(w) = \sum_{i=1}^{D-1} |w_{i+1} - w_i| = \|Cw\|_1.$$
(b) The TV-regularized regression problem can be written as
\[
\min_{w \in \mathbb{R}^D} \frac{1}{2} \|Xw - y\|_2^2 + \lambda \|Cw\|_1
\]  
(1)
or in summation form as
\[
\min_{w \in \mathbb{R}^D} \frac{1}{2} \sum_{i=1}^{N} (x_i^T w - y_i)^2 + \lambda \sum_{i=1}^{D-1} |w_{i+1} - w_i|.
\]

Determine the ADMM updates to solve Eq. (1). You may need to apply the splitting technique and introduce a new optimization variable.

c) Implement your devised ADMM algorithm by completing tvADMM.m in the files for download and test your algorithm on prob1.m. **Turn in:**

- Your tvADMM.m code
- A plot of the true and estimated functions
- A plot of the relative error between iterations as a function of the iteration number, where the error is defined as
\[
\text{err}_k = \frac{\|w^{(k)} - w^{(k-1)}\|}{\|w^{(k)}\|},
\]
where \(w^{(k)} \in \mathbb{R}^D\) denotes the estimate of \(w\) at the \(k\)th iteration.