CLOSED FORM FOURIER-BASED TRANSMIT BEAMFORMING FOR MIMO RADAR

John Lipor[*]   *Sajid Ahmed*[†]   *Mohamed-Slim Alouini*[†]

[*]University of Michigan, Ann Arbor
[†]King Abdullah University of Science and Technology
lipor@umich.edu, {sajid.ahmed,slim.alouini}@kaust.edu.sa

ABSTRACT

In multiple-input multiple-output (MIMO) radar setting, it is often desirable to design correlated waveforms such that power is transmitted only to a given set of locations, a process known as beampattern design. To design desired beampattern, current research uses iterative algorithms, first to synthesize the waveform covariance matrix, $\mathbf{R}$, then to design the actual waveforms to realize $\mathbf{R}$. In contrast to this, we present a closed form method to design $\mathbf{R}$ that exploits discrete Fourier transform and Toeplitz matrix. The resulting covariance matrix fulfills the practical constraints and performance is similar to that of iterative methods. Next, we present a radar architecture for the desired beampattern that does not require the synthesis of covariance matrix nor the design of correlated waveforms.

Index Terms— Beampattern, discrete Fourier transform (DFT), multiple-input-multiple-output (MIMO) radar, waveform design.

1. INTRODUCTION

Multiple-input multiple-output (MIMO) radar has been a topic of interest for researchers in recent years due to the performance advantages offered over phased-array radar (see [1] and the references therein). In the traditional phased-array setup, each transmit antenna transmits a phase-shifted version of the same baseband waveform. In this case, the probing signals are fully correlated and the transmit covariance matrix is equal everywhere, resulting in a narrow beam focused at a single point. A common MIMO setup transmits mutually orthogonal waveforms from each antenna. Here the transmit covariance matrix is equal to the identity matrix, and equal power is transmitted in all directions. The topic of transmit beampattern design lies between these two extremes and is the subject of this paper.

In the transmit beampattern design problem, the user wishes to transmit power exclusively to one or more prespecified regions of interest (ROIs). Previous work on beampattern

design relies largely on iterative methods of solution [2–4], which can incur a large computational complexity and prevent the use of these methods in real-time beamforming applications. Closed-form solutions exist [3, 5], but often fail to fulfill practical constraints such as having uniform diagonal elements. Once the covariance matrix has been designed, the next step is to design the actual waveforms to be transmitted that have the desired cross-correlations [6–8]. This step poses another difficult problem, where iterative methods are again required. Moreover, the resulting beampatterns suffer from high sidelobe levels.

This paper involves two contributions. First, we present a novel closed-form method to synthesize the covariance matrix for the desired beampattern under the setting of a uniform linear array (ULA). The method relies on the simple procedure of choosing discrete Fourier transform (DFT) coefficients to have nonzero values, and then generating a Toeplitz matrix based on the corresponding "discrete time signal". This process is similar to frequency sampling filter design. Second, we demonstrate how the method can be used to design the actual transmit waveforms in a manner similar to [9] and provide an analysis of the resulting peak-to-average-power ratio (PAPR).

2. PROBLEM FORMULATION

Consider a MIMO radar system with N_T co-located transmit antennas in a ULA having interelement spacing d and transmission wavelength λ. Define the transmitted baseband signal vector as $\mathbf{x}(n) = \begin{bmatrix} x_1(n) & x_2(n) & ... & x_{N_T}(n) \end{bmatrix}^T$, where $(\cdot)^T$ denotes the transpose. Assuming the transmitted probing signals are narrowband and that the propagation is nondispersive, the signal received by a target located at an angle θ at time n can be written as $r(n; \theta) = \mathbf{a}_T^H(\theta)\mathbf{x}(n)$, where $(\cdot)^H$ denotes the conjugate transpose and $\mathbf{a}_T(\theta) = \begin{bmatrix} 1 & e^{-j\frac{2\pi d}{\lambda}\sin(\theta)} & ... & e^{-j\frac{2(N_T-1)\pi d}{\lambda}\sin(\theta)} \end{bmatrix}^T$ represents the transmit steering vector. The transmitted average power at location θ can then be found as

$$P(\theta) = \mathbf{a}_T^H(\theta)\mathbf{R}\mathbf{a}_T(\theta), \qquad (1)$$

where $\mathbf{R}$ is the covariance matrix of the transmitted waveforms. The objective is to design $\mathbf{R}$ such that the transmitted power matches some desired beampattern as closely as possible while fulfilling the constraints that $\mathbf{R}$ be positive semidefinite and constant along the main diagonal (uniform elemental power).

Previous work utilizes optimization techniques to minimize a variety of cost functions. The work in [2,3] formulates this problem as a polynomial time semidefinite quadratic program (SQP), where the cost function corresponds to the least-squares minimization problem

$$J(\mathbf{R}) = \sum_{l=1}^{L} \left(\mathbf{a}_T^H(\theta_l)\mathbf{R}\mathbf{a}_T(\theta_l) - \alpha P_d(\theta_l) \right)^2, \qquad (2)$$

where $P_d(\theta_l)$ is the desired beampattern defined over the grid points $\{\theta_l\}_{l=1}^{L}$, and α is a scaling factor. Minimization of this cost function can be achieved using freely available software [10, 11], where the constraints can be trivially added.

3. COVARIANCE MATRIX DESIGN

In this section, we describe the proposed method of covariance matrix design, which exploits the DFT. By specifying a rectangular window (or set of windows) of varying width in the frequency domain, the user can equivalently achieve beampatterns of varying widths in the spatial domain. The method is similar to the familiar frequency sampling method for finite impulse response (FIR) filter design, which was noted as a method of receive beamforming in [12]. This further reinforces the similarities between filter design and MIMO radar transmit beamforming described in [13].

To establish the relationship between the rectangular window in the frequency domain and the desired beampattern, let $\{H(k)\}_{k=0}^{N_T-1}$ denote the spatial-frequency domain samples and $\{h(n)\}_{n=0}^{N_T-1}$ be the corresponding space domain samples, in analogy with time domain samples, obtained after applying the inverse DFT. Using the DFT relationship, the following lemmas can be obtained. The proofs of these lemmas can be found in [14] and are omitted due to space constraints.

Lemma 1 *If $H(k)$ is real and an element of the set $\{0, 1\}$ for all k and $\mathbf{R}$ is the Toeplitz matrix formed using the samples $\{h(n)\}$ as*

$$\mathbf{R} = \begin{bmatrix} h(0) & h(1) & \cdots & h(N_T-1) \\ h^*(1) & h(0) & \cdots & h(N_T-2) \\ h^*(2) & h^*(1) & \cdots & h(N_T-3) \\ \vdots & \vdots & \ddots & \vdots \\ h^*(N_T-1) & h^*(N_T-2) & \cdots & h(0) \end{bmatrix}, \qquad (3)$$

then it can be easily proved that $\mathbf{R}$ will be positive semidefinite with maximum amplitude along the diagonal.

This lemma demonstrates that the described covariance matrix fulfills the required constraints.

Lemma 2 *If $H(k)$ is real and $\mathbf{R}$ is the Toeplitz matrix formed using the samples $\{h(n)\}$ as in Lemma 1, then it can be easily proved that*

$$\mathbf{e}^H(k)\mathbf{R}\mathbf{e}(k) = N_T H(k), \qquad (4)$$

where

$$\mathbf{e}(k) = \begin{bmatrix} 1 & e^{-j\frac{2\pi k}{N_T}} & \cdots & e^{-j\frac{2\pi k(N_T-1)}{N_T}} \end{bmatrix}^T \qquad (5)$$

is the Fourier vector corresponding to frequency k.

It is interesting to note that (4) is the frequency-domain equivalent to (2). This suggests a mapping between the frequency-domain window design and the resulting transmit beampttern, which is described in what follows. In order to map the discrete frequency point, k, onto the positive and negative spatial locations of θ_k, note that k denotes both positive and negative frequency points. The following relationship can be used to map the frequency components onto the spatial domain for an even number of transmit antennas

$$\theta_k = \begin{cases} \sin^{-1}\left(\frac{\lambda k}{dN_T}\right), & k = 0, \ldots, \frac{N_T}{2} \\ -\sin^{-1}\left(\frac{\lambda(N_T-k)}{dN_T}\right), & k = \frac{N_T}{2}+1, \ldots, N_T-1 \end{cases}. \qquad (6)$$

The mapping for an odd number of transmit antennas is similar. Given this mapping, it can be seen that a window in the frequency (k) domain results in a window of proportional width in the spatial (θ) domain. In the two extremes of a single nonzero point at $k = 0$ and a vector of all non-zero points, the beampattern results in the phased-array and omni-directional patterns, respectively.

While the beampattern matches (4) for spatial values corresponding to integer values of k, the function does not describe the beampattern for other values of θ. We now demonstrate that choosing the positive coefficients in the window function is equivalent to adding phased-array beams with centers at the given locations. We begin by noting that a covariance matrix to generate P phased-array beams with centers located at $\{\theta_i\}_{i=1}^{P}$, can be designed by creating a Toeplitz matrix as in (3) using the vector

$$\tilde{\mathbf{a}} = \sum_{i=1}^{P} \mathbf{a}_T^*(\theta_i) \qquad (7)$$

to define the first row. For the frequency-domain window with P unity coefficients at locations $\{p_i\}_{i=1}^{P}$, the transformed coefficients are

$$h(n) = \frac{1}{N_T} \sum_{i=1}^{P} e^{j2\pi p_i n/N_T}$$

$$= \frac{1}{N_T} \sum_{i=1}^{P} e^{j\frac{2\pi dn}{\lambda}\sin(\theta_i)}, \qquad (8)$$
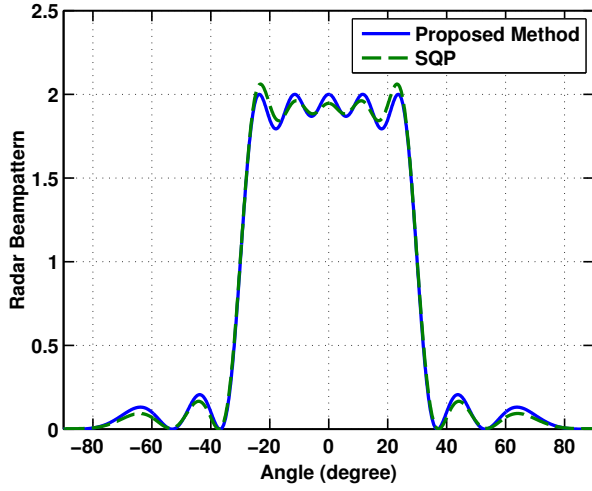
Fig. 1. Transmit beampatterns achieved by proposed method of covariance matrix design with $H(k) = 1, \ k = 0, 1, 2, 8, 9$ and SQP method. The ROI is $\theta \in [-30°, 30°]$ and $N_T = 10$.

where $\{\theta_i\}_{i=1}^P$ is the set of locations corresponding to $\{p_i\}_{i=1}^P$ and is found by solving (6). Vectorizing over all values of n, we obtain the vector $\mathbf{h}$ which corresponds to a scaled version of (7). For this reason, we conclude that the method of choosing beam locations using P DFT coefficients is equivalent to a sum of P phased array beams. Since the DFT coefficients represent mutually orthogonal frequencies, the resulting beams are placed at the nulls of the other beams in the sum, resulting in a smooth function within the window and overlapping sidelobes and nulls outside the ROI. In order to choose the values of k that define $\{p_i\}_{i=1}^P$ for a given ROI, Algorithm 1 can be employed.

Algorithm 1 Method of choosing $\{p_i\}_{i=1}^P$ for a given ROI

---

input: $P_d(\theta)$
$\theta_{max} = \max\{\theta \in \text{ROI}\}$
$\theta_{min} = \min\{\theta \in \text{ROI}\}$
solve (6) to obtain $k_{\max}$ corresponding to θ_{max} and $k_{\min}$ corresponding to θ_{min}
$k^+ \leftarrow \lfloor k_{\max} \rfloor$
$k^- \leftarrow \lceil k_{\min} \rceil$
$\{p_i\} = \{k \in \mathbb{Z} \mid k^- \le k \le k^+\}$

The main drawback of the method as proposed is that there are only N_T degrees of freedom available for beampattern design. Therefore, arrays with a low number of transmit antennas will only be able to transmit beams with a limited number of widths. However, with recent advances in sensor technology, the number of transmit antennas in systems has grown significantly. In such systems, the achievable resolution will be sufficient.

4. TRANSMIT SIGNAL DESIGN

In this section, a method of directly designing transmit waveforms to achieve the beampattern which is obtained by synthesizing $\mathbf{R}$ in the previous section is described. This method is referred to as the multi-rank beamformer in [9]. Here, we demonstrate how the covariance matrix from the preceding section can be easily adapted to this architecture and provide an analysis of the resulting PAPR using this method. Note that this method can be used for any rank-P $\mathbf{R}$ by decomposing into $\mathbf{R} = \mathbf{H}^H\mathbf{H}$, where $\mathbf{H} = \begin{bmatrix} \mathbf{h}_{p_1} & \mathbf{h}_{p_2} & \dots & \mathbf{h}_{p_p} \end{bmatrix}$.

Consider an arbitrary $H(k)$ used to define a transmit window and its corresponding $h(n)$. The beampattern as defined in the previous section can be achieved by transmitting a combination of P orthogonal sets of symbols drawn from any modulation scheme (e.g., BPSK or QPSK). Let $\mathbf{h}_{p_i}$ denote the vector of N_T elements defined by $h_{p_i}(m) = \frac{1}{N_T}e^{j2\pi p_i m/N_T}$, where $m = 0, \dots, N_T - 1$. Let $x_i(n)$ represent the symbol at time n that is weighted by the vector $\mathbf{h}_{p_i}$. The proposed architecture transmits the following signal from antenna m at time n, which is a weighted summation of orthogonal waveforms

$$v_m(n) = \sum_{i=1}^P x_i(n)h_{p_i}(m). \qquad (9)$$

An important practical consideration is that of PAPR, which is defined for antenna m by $\text{PAPR}(m) = \frac{\max_n P_{inst}(m;n)}{P_{avg}(m)}$. The average power transmitted from antenna m is

$$P_{avg}(m) = \mathrm{E}\{v_m(n)v_m^*(n)\} = \sum_{i=1}^P h_{p_i}(m)h_{p_i}^*(m) = \frac{P}{N_T^2}, \qquad (10)$$

where we have set $H(k) = 1$ for $k \in \{p_i\}_{i=1}^P$, and assumed the transmitted symbols to be orthonormal. When transmitting BPSK symbols, the peak instantaneous power occurs when all symbols are equal to one, resulting in

$$P_{peak}(m) = \frac{1}{N_T^2}\sum_{i=1}^P\sum_{l=1}^P e^{j\frac{2\pi m(p_i - p_l)}{N_T}}. \qquad (11)$$

From (11), it can be seen that the maximum instantaneous power occurs when $m = N_T$, in which case the instantaneous power is P^2/N_T^2 and the resulting PAPR is P. However, for $m \ne N_T$, the PAPR is less than P. The multi-rank beamformer in [9] encounters the same problem, and the reduction of PAPR for this method is an important topic for future research.

The proposed system provides benefits over the waveform design methods presented in [6–8] in that it does not require the generation of partially correlated symbols and therefore has a much lower computational cost. Rather, the transmitted signals can be designed directly. In addition, transmission of truly orthogonal signals (e.g., BPSK symbols drawn from
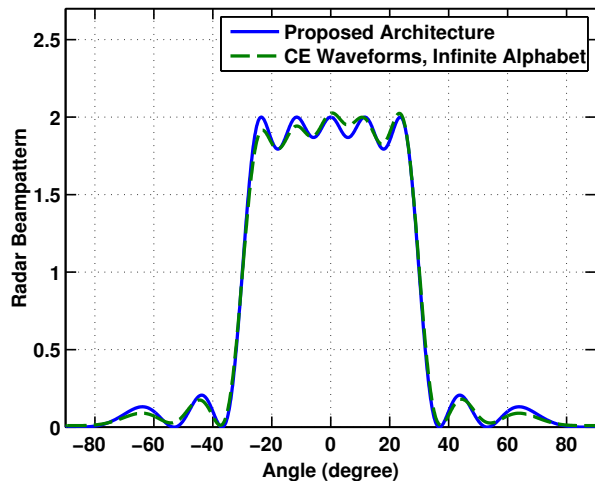
Fig. 2. Beampatterns achieved by transmitting symbols via the proposed architecture and the CE waveforms from [6].

Fig. 3. Receive beampatterns after applying the MVDR beamformer.

Hadamard code sequences) allows the transmitted beampattern to match the theoretical beampattern achieved by the covariance design procedure.

5. SIMULATION RESULTS

Simulations assume a uniform linear array with half-wavelength interelement spacing, total transmit power equal to one, and a mesh grid with spacing of $0.1°$. We begin by demonstrating the performance of the method for covariance matrix design presented in Section 3. Fig. 1 shows the resulting beampattern with $N_T = 10$ transmit antennas for the ROI defined by $\theta \in [-30°, 30°]$. Employing Algorithm 1, we obtain $\{p_i\} = \{0, 1, 2, 8, 9\}$ and set $H(k) = 1$ for these values of k. The solution found using the SQP as presented in [2] is also included for comparison. Define the mean-squared error (MSE) as $\frac{1}{L} \sum_{l=1}^{L} [P(\theta_l) - \alpha\phi(\theta_l)]^2$, where α is found using the SQP method. The resulting MSEs are 0.0322 for the proposed method and 0.0311 for the SQP method. We conclude that the proposed method provides performance which is comparable to that achieved by iterative methods while incurring a fraction of the computational cost. Similar results hold for different beampatterns and values of N_T.

Next, we demonstrate the performance achieved by the proposed radar architecture in Section 4. Fig. 2 shows the beampattern generated using the proposed method with orthogonal BPSK symbols drawn from Hadamard code sequences of length 128. The resulting beampattern formed using the waveforms described in [6] with PAPR = 1 is included for comparison. The desired beampattern is the same as in Fig. 1, and the resulting MSEs obtained by averaging over 100 Monte Carlo trials are 0.0322 for the proposed method and 0.0328 for the SQP. Thus, the proposed method provides a lower MSE and utilizes symbols from a finite
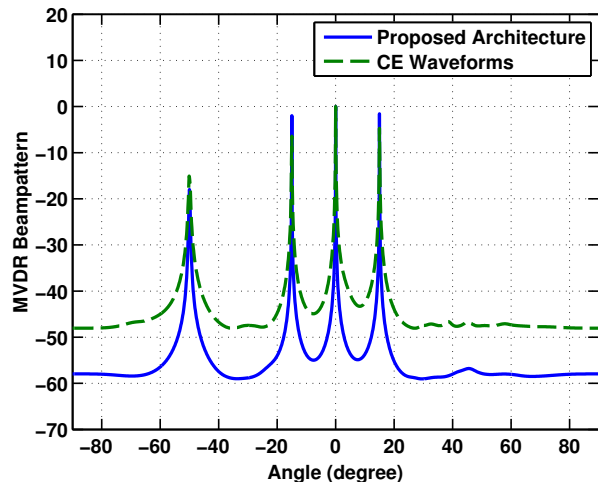
alphabet while incurring a much lower computational cost.

We conclude by demonstrating the receive beampattern after applying the minimum variance distortionless response (MVDR) beamformer. Consider the scenario in which there are three targets of interest located at $0°$ and $\pm15°$ and one interfering target located at $-50°$ that we wish to suppress. Probing signals are transmitted using the proposed radar architecture and the method presented in [6] with the same desired beampattern as in Fig. 1. The received signal is corrupted by white Gaussian noise with a variance 0.01. Fig. 3 shows the average resulting beampatterns from the two methods after transmitting 16 symbols over 1000 Monte Carlo simulations. Due to the reliance on perfectly uncorrelated symbols, the proposed method results in lower sidelobe levels as well as greater attenuation of the interfering target (about 3 dB in this case). Note that the CE waveforms also incur a downward bias of 6.3 dB for the target located at $-15°$ and 4.93 dB for the target located at $15°$. In contrast, the proposed architecture results in biases of 1.58 dB and 1.15 dB. Although not pictured, similar results are obtained for 256 transmitted symbols, as well as with higher noise variance.

6. CONCLUSION

We have demonstrated a closed-form method of covariance matrix design that exploits the properties of DFT coefficients. The resulting covariance matrix fulfills practical constraints, is computationally efficient, and results in performance similar to iterative solutions. We have also demonstrated a radar architecture that can be used with a set of orthogonal signals to match the desired beampattern. The resulting beampattern matches the theoretical pattern exactly, resulting in superior MSE performance compared to existing methods with significantly decreased computational burdens.

7. REFERENCES

[1] J. Li and P. Stoica, Eds., *MIMO Radar Signal Processing*. Hoboken, NJ: John Wiley & Sons, 2009.

[2] P. Stoica, J. Li, and Y. Xie, "On probing signal design for MIMO Radar," *IEEE Trans. Signal Process.*, vol. 55, no. 8, pp. 4151–4161, Aug. 2007.

[3] G. Ha and S. S. Abeysekera, "Mimo radar transmit beampattern design with ripple and transition band control," *IEEE Trans. Signal Process.*, to be published.

[4] S. Ahmed, J. Thompson, Y. Petillot, and B. Mulgrew, "Unconstrained synthesis of covariance matrix for MIMO Radar transmit beampattern," *IEEE Trans. Signal Process.*, vol. 59, no. 8, pp. 3837–3849, Aug. 2011.

[5] T. Aittomaki and V. Koivunen, "Signal covariance matrix optimization for transmit beamforming in MIMO Radars," in *Proc. Asilomar Conference on Signals, Systems, and Computers*, Nov. 2007, pp. 182–186.

[6] J. Li and P. Stoica, "Waveform synthesis for diversity-based transmit beampattern design," *IEEE Trans. Signal Process.*, vol. 56, no. 6, pp. 3461–3464, Jun. 2008.

[7] S. Ahmed, J. Thompson, Y. Petillot, and B. Mulgrew, "Finite alphabet constant-envelope waveform design for MIMO Radar transmit beampattern," *IEEE Trans. Signal Process.*, vol. 59, no. 11, pp. 5326–5337, Nov. 2011.

[8] S. Jardak, S. Ahmed, and M.-S. Alouini, "Generating correlated QPSK waveforms by exploiting real gaussian random variables," in *IEEE Asilomar Conference on Signal, System, and Computers*, 2012.

[9] B. Friedlander, "On transmit beamformign for MIMO radar," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 48, no. 4, pp. 3376–3388, Oct. 2012.

[10] M. Grand and S. Boyd, "CVX: Matlab software for disciplined convex programming, version 2.0 beta," http://cvxr.com/cvx, Jul. 2013.

[11] M. Grant and S. Boyd, "Graph implementations for nonsmooth convex programs," in *Recent Advances in Learning and Control*, ser. Lecture Notes in Control and Information Sciences, V. Blondel, S. Boyd, and H. Kimura, Eds. Springer-Verlag Limited, 2008, pp. 95–110.

[12] B. V. Veen and K. Buckley, "Beamforming: A versitile approach to spatial filtering," *IEEE ASSP Mag.*, vol. 5, no. 2, pp. 4–24, Apr. 1988.

[13] D. R. Fuhrmann, J. P. Browning, and M. Rangaswamy, "Signaling strategies for the hybrid MIMO phased-array Radar," *IEEE J. Sel. Topics Signal Process.*, vol. 4, no. 1, pp. 66–78, Feb. 2010.

[14] J. Lipor, S. Ahmed, and M.-S. Alouini, "Fourier-based transmit beampattern design using mimo radar," *IEEE Trans. Signal Process.*, 2014, 10.1109/TSP.2014.2307838.