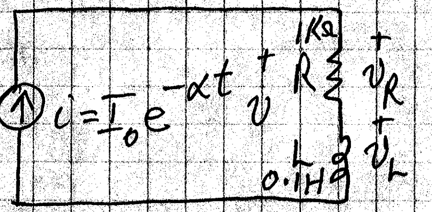
ECE 241 Lecture 9 Chapter 5 Forced Response

Response to exponential forcing function:

Exercise 5.1



(a) Circuit time constant $\tau = L/R = 0.1/10^3 = 0.1\text{ms}$

(b) v for $i = 2e^{-5000t}$ A $\qquad v = iR + L\frac{di}{dt}$

$$v = 10^3 \, 2e^{-5000t} + 0.1(2e^{-5000t})(-5000)$$
$$= 1000\, e^{-5000t}$$

(c) v for $i = 2e^{-20,000t} \rightarrow v = 10^3\, 2e^{-20,000t} + 0.1(2e^{-20,000t})(-20,000)$

$$= -2000\, e^{-20,000t}$$

DC Response

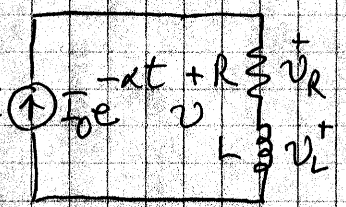
$s = 0 \quad i = I_0 e^{0t} = I_0 \leftarrow$ cap's $\rightarrow$ DC

$Z_R(0) = R \qquad Z_L(0) = 0 \qquad Z_C(0) = \infty$

$\qquad (= sL) \qquad\qquad (= 1/sC)$

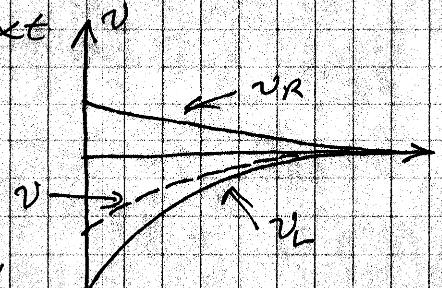
S.C. at DC $\qquad$ O.C. at DC

Exponential Forcing Function



$v = v_R + v_L = (R - \alpha L) I_0 e^{-\alpha t}$

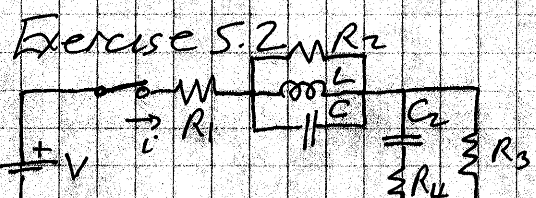
Voltage response also exponential of time constant (shape) as the forcing function



Note: +ve $i \rightarrow$ −ve v due to inductance L. (opposes change in current)

Note: if $\alpha = R/L$ i.e. forcing time constant = circuit time constant then $v = 0$

Exercise 5.2



$V = 5\text{v} \quad R_1 = R_2 = 1\text{K}\Omega$

$R_3 = R_4 = 4\text{K}\Omega$

$L = 1\text{mH}$

$C_1 = C_2 = 10\mu\text{F}$

S closed for long time

(a) $i = V/(R_1 + R_3) = 5\text{v}/5\text{K} = 1\text{mA}$

(b) Voltage across resistors:

$v_{R1} = i \times 1\text{K} = 1\text{v}$

$v_{R2} = 0 \quad$ (L is s.c.)

$v_{R3} = i \times 4\text{K} = 4\text{v}$

$v_{R4} = 0 \quad$ (C is o.c.)

SINUSOIDS : PHASOR REPRESENTATION

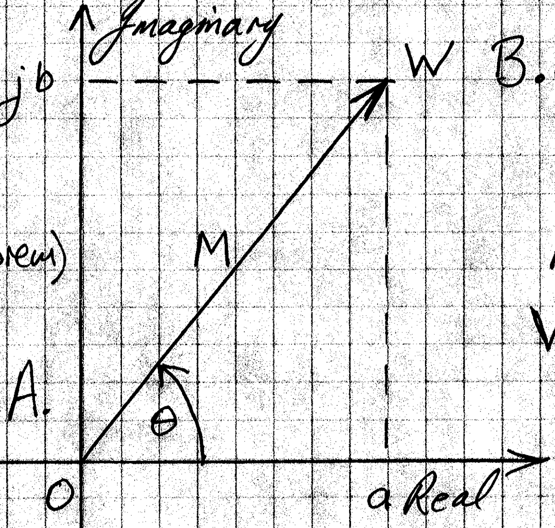
vector (boldface in text)

A. $\mathbf{W} = a + jb$

$= M(\cos\theta + j\sin\theta)$

$= M e^{j\theta}$ (Euler's theorem)

$= M\angle\theta$



B. Rotating vector angular velocity ω

i.e. $\theta \longrightarrow \omega t + \theta$

$\mathbf{W}(t) = Me^{j(\omega t + \theta)}$

Rectangular $\longrightarrow$ Polar

$M = (a^2 + b^2)^{1/2}$

$\theta = \arctan b/a$

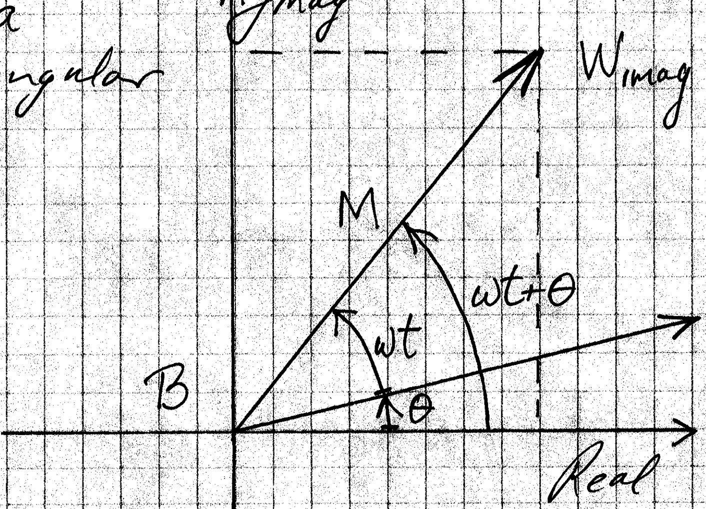
Polar $\longrightarrow$ Rectangular

$a = M\cos\theta$

$b = M\sin\theta$

$W_{real} = M\cos(\omega t + \theta)$

$W_{imag} = M\sin(\omega t + \theta)$



Represent sinusoids by W_{real} (or W_{imag})

Exercise 5.4

(a) $\mathbf{W}_1 = 5 + j5$ $\quad \mathbf{W}_2 = 5 - 5j$ convert to polar

$M_1 = (5^2 + 5^2)^{1/2} = 5\sqrt{2}$ $\qquad M_2 = (5^2 + (-5)^2)^{1/2} = 5\sqrt{2}$

$\theta_1 = \arctan 5/5 = \pi/4$ $\qquad \theta_2 = \arctan \frac{-5}{5} = -\pi/4$

$\therefore \mathbf{W}_1 = 5\sqrt{2}\, e^{j\pi/4}$ $\qquad \mathbf{W}_2 = 5\sqrt{2}\, e^{-j\pi/4}$

(b) Convert $\mathbf{W}_3 = e^{j\pi/3}$ $\quad \mathbf{W}_4 = e^{j4\pi/3}$ to rectangular

$a_3 = 1\cos\pi/3 = 0.5$ $\qquad a_4 = 1\cos\frac{4\pi}{3} = -\cos\pi/3 = -0.5$

$b_3 = 1\sin\pi/3 = \sqrt{3}/2$ $\qquad b_4 = 1\sin\frac{4\pi}{3} = -\sin\pi/3 = -.866$

$\therefore \mathbf{W}_3 = 0.5 + j0.866$ $\qquad \therefore \mathbf{W}_4 = -0.5 - j0.866$

For $v(t) = V_m \cos(\omega t + \theta) = \sqrt{2}\, V \cos(\omega t + \theta)$

V_m ← peak, V ← rms

$= \mathrm{Re}\{V_m e^{j(\omega t + \theta)}\} = \mathrm{Re}\{\sqrt{2}\,\underbrace{(V e^{j\theta})}_{\text{"Phasor"}}\, e^{j\omega t}\}$

"Phasor" $\mathbf{V} = V e^{j\theta}$ ← vector

$\mathbf{V}$ is a "transform" of $v(t)$

$e^{j\omega t}$ is common to all terms
and can be removed for calculation
with the understanding that it can be restored later.

Exercise 5-5:
For $v_1(t) = \mathrm{Re}\{5 e^{j(100\pi t + \pi/4)}\}$ & $v_2(t) = \mathrm{Re}\{10 e^{j(100\pi t + 1.21)}\}$

(a) Effective values:

$V_{eff_1} = \frac{5}{\sqrt{2}} = 3.54\,\mathrm{v}$ $\quad V_{eff_2} = \frac{10}{\sqrt{2}} = 7.07\,\mathrm{v}$

(b) Rectangular forms of phasors $Ve^{j\theta}$

$\mathbf{V}_1 = 3.54 \cos \pi/4 + j3.54 \sin \pi/4$
$= 2.5 + j\,2.5$

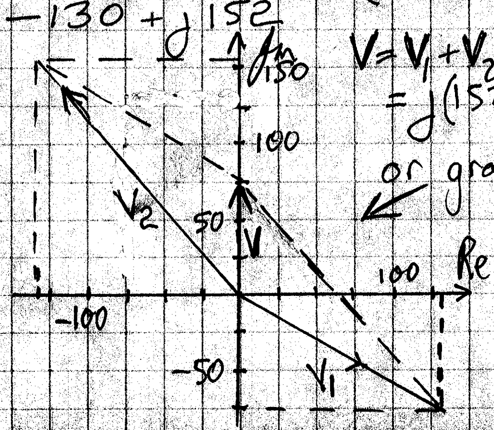
$\mathbf{V}_2 = 7.07 \cos 1.21 + j\,7.07 \sin 1.21$
$= 2.5 + j\,6.6$

Exercise 5-6:
Represent $i = 5 \sin(100t + 120°)$ as a phasor
$= 5 \cos(100t + 120° - 90°)$
$= 5 \cos(100t + 30°)$

$\mathbf{I} = \frac{5}{\sqrt{2}} \angle 30°$
$= 3.54 \angle 30°$
$= 3.54\, e^{j\pi/6}$

Exercise 5-7:
$v_1 = 150\sqrt{2} \cos(377t - \pi/6)$
$\therefore \mathbf{V}_1 = 130 - j\,75$
$\mathbf{V}_2 = 200 \angle 130.5° = 200(\cos 130.5° + j \sin 130.5°)$
$= -130 + j\,152$

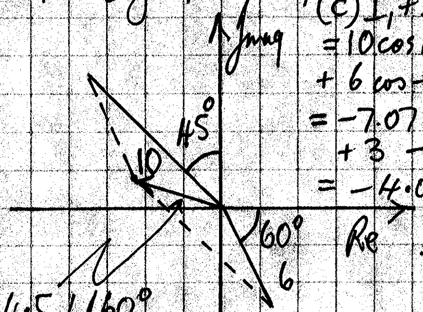
$\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2$
$= j(152 - 75) = j77$
or graphically



Exercise 5-8: $\begin{cases} i_1 = 10\sqrt{2} \cos(\omega t + 135°) \\ i_2 = 6\sqrt{2} \cos(\omega t - 60°) \end{cases}$

(a) As phasors $\mathbf{I}_1 = 10 \angle 135°$ $\quad \mathbf{I}_2 = 6 \angle -60°$

(b) $\mathbf{I}_1 + \mathbf{I}_2$ graphically



(c) $\mathbf{I}_1 + \mathbf{I}_2$
$= 10 \cos 135° + j \sin 135°$
$+ 6 \cos -60° + j \sin -60°$
$= -7.07 + j\,7.07$
$+ 3 - j\,5.2$
$= -4.07 + j\,1.87$
$= 4.48 \angle 155.3°$

AC CIRCUIT ANALYSIS

Assume "AC" is sinusoidal, unless stated otherwise.

Sinusoidal $\leftrightarrow$ exponential $\leftrightarrow$ phasor representations

Current $i = \sqrt{2} I \cos\omega t = I\angle 0^\circ$ in R

$$v_R = iR = \sqrt{2}\, IR \cos\omega t = \sqrt{2} V_R \cos\omega t = V_R\angle 0^\circ$$

Impedance $\dfrac{V_R\angle 0^\circ}{I\angle 0^\circ} = \dfrac{R I\angle 0^\circ}{I\angle 0^\circ} = R\angle 0^\circ \leftarrow$ Num$\angle$ − Deno$\angle$

Current — — — — — — — — in L

$$v_L = L\frac{di}{dt} = \sqrt{2}\,\omega L I(-\sin\omega t) = \sqrt{2}\,\omega L I \cos(\omega t + 90^\circ)$$

Impedance $\dfrac{V_L\angle 90^\circ}{I\angle 0^\circ} = \dfrac{\omega L I\angle 90^\circ}{I\angle 0^\circ} = \omega L\angle 90^\circ = j\omega L$

j implies 90° anticlockwise rotation

Voltage $v = \sqrt{2} V \cos\omega t = V\angle 0^\circ$ across C

$$i_C = C\frac{dv}{dt} = \sqrt{2}\,\omega C V(-\sin\omega t) = \sqrt{2}\,\omega C V(\cos \omega t + 90^\circ)$$

Impedance $\dfrac{V\angle 0^\circ}{\omega C V\angle 90^\circ} = \dfrac{1}{\omega C}\angle -90^\circ = \dfrac{-j}{\omega C} \leftarrow -j$ clockwise rotation 90°

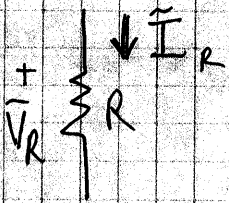
Resistance $\mathbf{Z}_R = Z_R(j\omega) = R = R\angle 0^\circ$

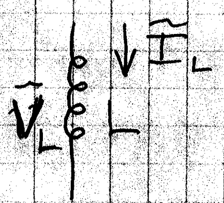
Inductance $\mathbf{Z}_L = Z_L(j\omega) = j\omega L = \omega L\angle 90^\circ$

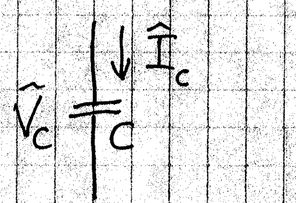
Capacitance $\mathbf{Z}_C = Z_C(j\omega) = \dfrac{1}{j\omega C} = \dfrac{-j}{\omega C} = \dfrac{1}{\omega C}\angle -90^\circ$

General term Impedance $\mathbf{Z} = R + jX$

Reactance







$$\hat{V}_R = \hat{I}_R R$$

$$\hat{I}_R = \frac{1}{R}\tilde{V}_R = G\hat{V}_R$$

Conductance

$$\hat{V}_L = j\omega L \hat{I}_L$$

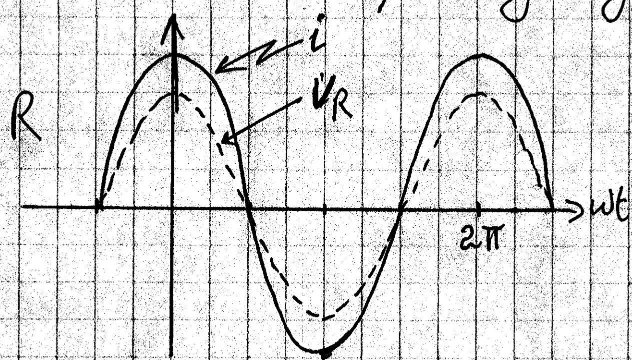
$$\hat{I}_L = \frac{1}{j\omega L}\hat{V}_L$$

$$\hat{V}_C = \frac{1}{j\omega C}\hat{I}_C$$

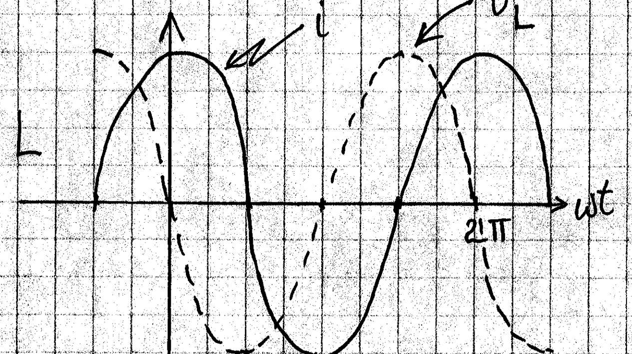
$$\hat{I}_C = j\omega C \tilde{V}_C$$

Susceptance

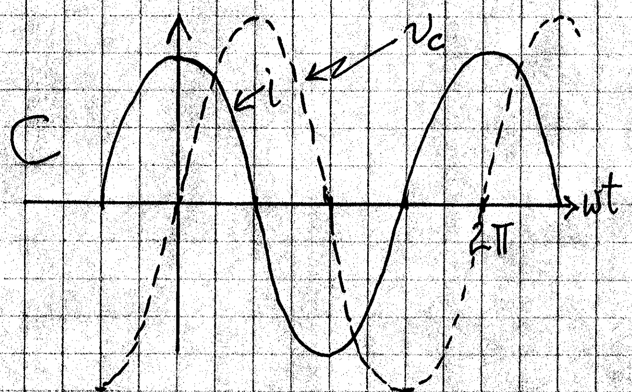
Admittance $\tilde{Y} = \tilde{G} + j\tilde{B}$



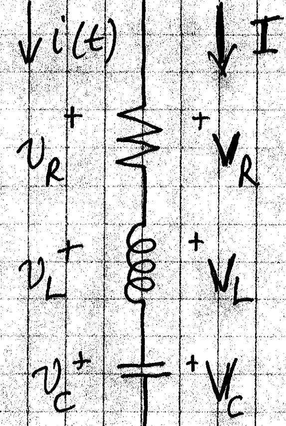
voltage in phase with current

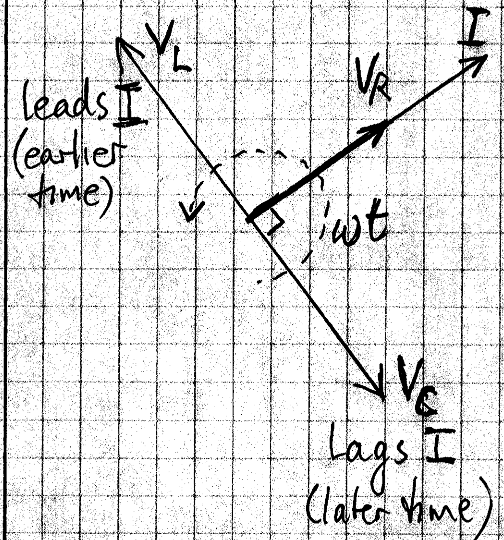


Voltage LEADS current by $\pi/2$



Voltage LAGS current by $\pi/2$





KIRCHOFF'S LAWS hold for phasors

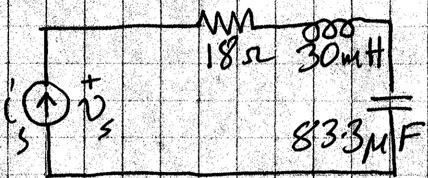
$$\sum \mathbb{V} = 0 = \mathbb{V}_1 + \mathbb{V}_2 + \mathbb{V}_3 + \cdots \mathbb{V}_n$$

around any loop

$$\sum \mathbb{I} = 0 = \mathbb{I}_1 + \mathbb{I}_2 + \mathbb{I}_3 + \cdots \mathbb{I}_n$$

into any node

Exercise 5.9



$i_s = 120\sqrt{2}\cos(1000t + 90°)$. Find v_s

$$\mathbb{Z} = 18 + j\omega\, 30\times10^{-3} + \frac{1}{j\omega\, 83.3\times10^{-6}}$$

$$\xrightarrow{\omega = 10^3} 18 + j30 - j\,10^3/83.3$$

$$= 18 + j30 - j12 = 18 + j18 = 18\sqrt{2}\angle 45°$$

$\mathbb{I}_s = 120\angle 90°$, so $\mathbb{V}_s = \mathbb{I}_s\mathbb{Z} = 120\times18\sqrt{2}\angle 90° + 45°$

$$= 3054\angle 135°$$

$\therefore v(t) = 3054\sqrt{2}\cos(1000t + 135°)$ volts.

In General — impedance $\mathbb{Z}$ is complex, and $\mathbb{V} = \mathbb{Z}\mathbb{I}$

where $\mathbb{Z} = Z\angle\phi_z = R + jX$

where $Z = R^2 + X^2 \qquad \phi_z = \arctan X/R$

$\mathbb{Z}$ complex but not a phasor, not varying with time.

Exercise 5.10

(a) Find impedance of 100mH coil of resistance 400Ω at 2000Hz.

$$\mathbb{Z} = 400 + j\,2000\times2\pi\times10^{-1} = 400 + j1257 = (400^2 + 1257^2)^{1/2}\angle \arctan\tfrac{1257}{400}$$

$$= 1319\angle 72.3°$$

(b) Find C in RC series ckt for $\mathbb{Z} = 100\angle -80°$ at 5KHz.

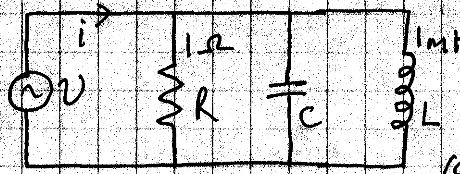
$$\mathbb{Z} = 100\angle -80° = 100\cos 80° - j\,100\sin 80°$$

$$\mathbb{Z}_{RC} = R - j\frac{1}{2\pi\, 5000\, C} \quad \therefore \frac{1}{\pi\, 10^4 C} = 100\times0.985 \quad \text{ie. } C = \frac{1}{10^6\pi(.985)} \approx .032\mu F$$

(c) Find freq for R = 200Ω in series with L = 10mH to have 45° phase angle

$$\mathbb{Z} = R + j\omega L = 200 + j\,2\pi f\, 10^{-2} \quad \text{ie. } \frac{2\pi f\, 10^{-2}}{200} = \tan 45° = 1 \quad \therefore f = \frac{20000}{2\pi} = 3.183\,\text{KHz}$$

Exercise 5.11 $\quad i = 10\sqrt{2}\cos(1000t + 45°)$



Find Z_{eq}, R_{eq}, X_{eq} for (a) $C = 0.5\,mF$
(b) $C = 2\,mF$

(a) $Z_{eq} = \dfrac{1}{\frac{1}{R} + \frac{1}{j\omega L} + j\omega C} = \dfrac{1}{1 - \frac{j}{10^{-3}10^{+3}} + j\frac{10^{-3}10^{3}}{2}}$

$= \dfrac{1}{1 - j0.5} = \dfrac{1}{1.11\angle -26.6°} = 0.9\angle 26.6°$

$= 0.82 + j.365$

$\underbrace{1.11}_{(1+.25)^{1/2}}\quad \underbrace{-26.6°}_{\arctan^{-}0.5}$

(b) $Z_{eq} = \dfrac{1}{1 - j + j\,10^{3}\,2\,10^{-3}} = \dfrac{1}{1+j} = \dfrac{1-j}{2} = \dfrac{\sqrt{2}}{2}\angle \arctan -1 = \dfrac{1}{\sqrt{2}}\angle -45°$

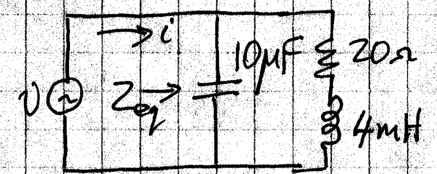
$= \dfrac{1}{\sqrt{2}}\cos -45 + j\dfrac{1}{\sqrt{2}}\sin -45°$

$= 0.5 - j0.5$

So (a) $\rightarrow R_{eq} = 0.82\,\Omega \quad X_{eq} = 0.365\,\Omega$

(b) $\rightarrow R_{eq} = 0.5\,\Omega \quad X_{eq} = -0.5\,\Omega$

Exercise 5.12 $\quad v = 12\sqrt{2}\cos 5000t$



Find Z_C, Z_L, Z_R, Z_{eq}, i

$Z_R = 20 + j0 = 20\angle 0°$

$Z_L = 0 + j\,5000 \times 4 \times 10^{-3} = 0 + j20 = 20\angle 90°$

$Z_C = 0 - j\dfrac{1}{5\times10^{3}\times10^{-5}} = 0 - j20 = 20\angle -90°$

$Z_{RL} = Z_R + Z_L = 20 + j0 + 0 + j20 = 20 + j20 = 20\sqrt{2}\angle 45°$

$= 28.3\angle 45°$

$\therefore Z_{eq} = \dfrac{Z_{RL} Z_C}{Z_{RL} + Z_C} = \dfrac{28.3 \times 20\angle 45° - 90°}{20 + j20 - j20} = 28.3\angle -45°$

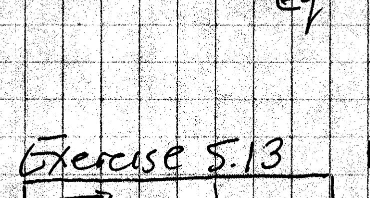
$= 20.01 - j20.01$

$\therefore I = \dfrac{V}{Z_{eq}} = \dfrac{12\angle 0°}{28.3\angle -45°} = 0.424\angle 45°$

$= 0.424\sqrt{2}\cos(5000t + 45°)$

$= 0.60\cos(5000t + 45°)$

Exercise 5.13 $\quad$ Find I_C, V_R From Example 5 $\quad Z_R = 20 \quad Z_C = -j20 \quad Z_L = j15$



$I_C = \dfrac{Z_{LR}}{Z_{LR} + Z_C} I = \dfrac{20 + j15}{20 + j15 - j20} \cdot 49\angle 39° = \dfrac{4+3j}{4-j} \cdot 49\angle 39°$

$= \dfrac{5\angle 36.9°}{4.12\angle -14.0}\, 0.49\angle 39° = 0.6\angle 89.9°$

$V_R = \dfrac{Z_R}{Z_R + Z_L} V = [20/(20 + j15)]\, 12\angle 0° = \dfrac{4}{5\angle 36.9}12 = 9.6\angle -36.9°$

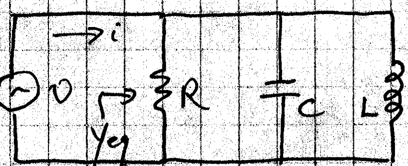
ADMITTANCE $\mathbf{Y} = \dfrac{\mathbf{I}}{\mathbf{V}} = \dfrac{1}{\mathbf{Z}}$

$$Y_R = \frac{1}{R\angle 0^\circ} = G\angle 0^\circ \qquad Y_L = \frac{1}{j\omega L} = \frac{1}{\omega L}\angle -90^\circ \qquad Y_C = j\omega C = \omega C\angle +90^\circ$$

$$\mathbf{Y} = Y\angle\phi_Y = G + jB \qquad Y = (G^2 + B^2)^{1/2}$$

$$\phi_Y = \arctan\frac{B}{G}$$

Exercise 5.14 $\quad v = 120\sqrt{2}\cos(1000t + 90^\circ)$



$R = 15\,\Omega \quad C = 83.3\,\mu F \quad L = 30\,mH$

Find $Y_R \quad Y_L \quad Y_C \quad Y_{eq}\,,\ I$

$$Y_L = \frac{1}{1000\ 30\ 10^{-3}}\angle -90^\circ = \frac{1}{30}\angle -90^\circ\ S$$

$$Y_R = \frac{1}{15}\ S \qquad Y_C = 1000 \times 83.3 \times 10^{-6}\angle +90^\circ\ S = .0833\angle +90^\circ\ S = \frac{1}{12}\angle +90^\circ\ S$$

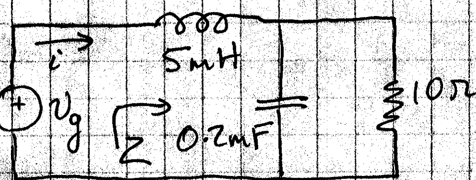
$$\therefore Y = Y_R + Y_C + Y_L = \frac{1}{15} + \frac{1}{12}\angle +90 + \frac{1}{30}\angle -90$$

$$= .067 + \left(\frac{1}{12}\times 0\right) + j.083 + \left(\frac{1}{30}\times 0\right) - j\,.033$$

$$= .067 + j\,.05 = .084\angle 36.9^\circ$$

so $I = \mathbf{Y}\mathbf{V} = 120\angle 90^\circ\,(.084)\angle 36.9^\circ = 10.08\angle 126.9^\circ \approx 10\angle 127^\circ\ A$

Exercise 5.15 $\quad v_g = 100\cos 1000t$



(a) $$Z = j\omega L + \frac{1}{\frac{1}{R} + j\omega C} = j10^3 \times 5\times 10^{-3} + \frac{1}{.1 + j10^3(.2)10^{-3}}$$

$$= 5j + \frac{10}{1 + 2j} = 5j + \frac{10(1 - 2j)}{1 + 4}$$

$$= 5j + 2(1 - 2j) = 2 + j \quad \left(\begin{array}{c}\text{incorrect}\\ \text{in text}\end{array}\right)$$

$$= \sqrt{5}\ \arctan 1/2$$
$$= 2.24\angle 26.56^\circ$$

(b) $$\therefore I = \frac{V}{Z} = \frac{100/\sqrt{2}\angle 0^\circ}{2.24\angle 26.56^\circ} = 31.6\angle -26.56^\circ$$

(c) $i(t) = \underset{\uparrow\ 44.7\,A}{31.6\sqrt{2}}\cos(1000t - 26.56^\circ)$

See Section 5.4 Analogs & Duals