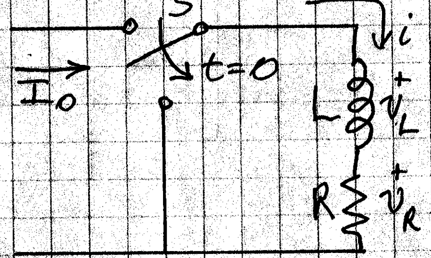
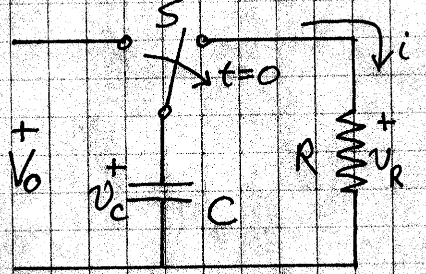
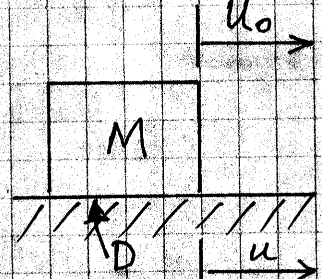
E241 Lecture 8 TRANSIENTS I (Chapter 4)

First Order Systems







Initially I_o in L, R
$i(0) = I_o$

Initially V_o across R, C
$v_C(0) = v_R(0) = V_o$

Initially velocity U_o
Remove force at $t=0$

For $t > 0$

$$v_L + v_R = 0 \qquad v_c = v_R \qquad M\frac{du}{dt} = -Du$$

$$L\frac{di}{dt} + iR = 0 \qquad V_o - \frac{1}{C}\int_0^t i\,dt = iR$$

Solutions:-

$$L\frac{di}{dt} + iR = 0 \qquad R\frac{di}{dt} + \frac{i}{C} = 0 \qquad M\frac{du}{dt} + Du = 0$$

Separate variables:

$$\frac{di}{i} = -\frac{R}{L}dt \qquad \frac{di}{i} = -\frac{1}{RC}dt \qquad \frac{du}{u} = -\frac{D}{M}dt$$

$$\int_{I_o}^{i}\frac{di}{i} = -\frac{R}{L}\int_0^t dt \qquad \int_{V_o/R}^{i}\frac{di}{i} = -\frac{1}{RC}\int_0^t dt \qquad \int_{U_o}^{u}\frac{du}{u} = -\frac{D}{M}\int_0^t dt$$

$$\ln i\Big]_{I_o}^{i} = -\frac{t}{(L/R)} \qquad \ln i\Big]_{V_o/R}^{i} = -\frac{t}{RC} \qquad \ln u\Big]_{U_o}^{u} = -\frac{t}{(M/D)}$$

$$i/I_o = \exp -\frac{t}{(L/R)} \qquad i/\left(\frac{V_o}{R}\right) = \exp -\frac{t}{RC} \qquad u/U_o = \exp -\frac{t}{(M/D)}$$

$$i(t) = I_o \exp -t/(L/R) \qquad i(t) = \frac{V_o}{R}\exp -\frac{t}{RC} \qquad u(t) = U_o \exp -\frac{t}{(M/D)}$$

Note text method — assumes solution of the form Ae^{st}

so for $L\frac{di}{dt} + Ri = 0$ & $i = Ae^{st}$

$$sLAe^{st} + RAe^{st} = 0$$

$$(R + sL)Ae^{st} = 0$$

so for $Ae^{st} \neq 0$, $s = -R/L$

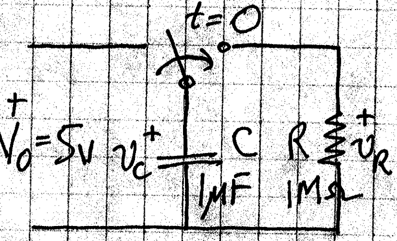
and $i = Ae^{-\frac{R}{L}t}$

When $t = 0^+$ $i(0) = I_o = A$

$\therefore$ $i(t) = I_o \exp - \frac{t}{(L/R)}$

(See text for RC example)

Exercise 4.1



(a) Voltage $v_R(0^+) = V_0 = 5V$

(b) Hence $v_R(t) = V_0 \exp - \frac{t}{RC}$

$= 5 \exp -t$

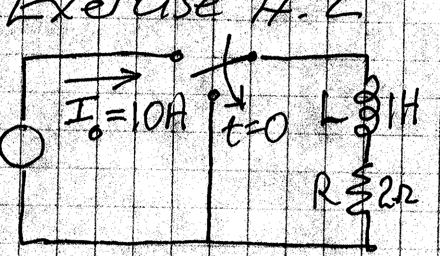
$v_R(t) = \frac{V_0}{2}$ when $\exp -t = 0.5$

i.e. at $t = \ln 2 = 0.693s$

Current $i(0.693) = \frac{V_0/2}{R} = \frac{2.5V}{1M\Omega} = 2.5\mu A$

[Energy stored in C dissipated in R.]

Exercise 4.2



$i(t) = I_o \exp - \frac{t}{(L/R)} = 10 \exp -2t$

At $t = 1s$, $i(1) = 10 \exp -2 = 1.35A$

At $t = 10s$ $i(10) = 10 \exp -20 = 20.6 \times 10^{-9} A$

$= 20.6 nA$

Procedure for solution:

1. Write governing equation (Kirchoff)
2. Convert to homogeneous differential equation
3. Assume a general exponential solution
4. Find exponenents from homogeneous equation
5. Find coefficients from initial conditions.

Note: $\exp -\lambda$ where λ is a constant.

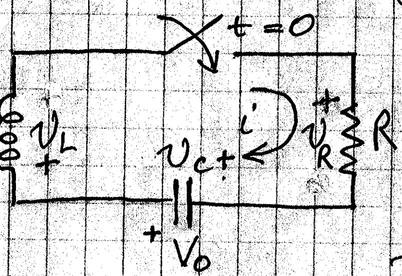
So — solutions $\exp -\frac{t}{RC}$, $\exp -\frac{t}{(L/R)}$, $\exp -\frac{t}{(M/D)}$

are of the form $\exp -t/\tau$

where τ is the "time constant"

e.g. $[RC] = [R][C] = \frac{[V]}{[I]} \cdot \frac{[I]}{[V]/[T]} = [T]$

Second Order Systems (RLC)



$$\sum v = 0 = v_R + v_C + v_L$$

$$= iR + \frac{1}{C}\int_0^t i\,dt - V_0 + L\frac{di}{dt}$$

Differentiate: —

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = 0$$

Assume a solution of the form $i = Ae^{st}$

$$\therefore \quad s^2 LAe^{st} + sRAe^{st} + \frac{1}{C}Ae^{st} = 0$$

which is (for $Ae^{st} \neq 0$) the Characteristic Equation

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

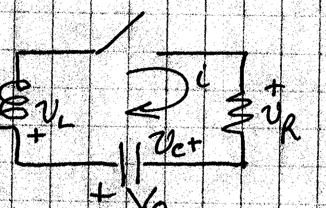
with roots $\quad s_1, s_2 = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$

Note that if both $i_1 = A_1 e^{s_1 t}$ and $i_2 = A_2 e^{s_2 t}$ satisfy the differential equation, then so does

$$i = i_1 + i_2 = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

where s_1, s_2 found as above, A_1 & A_2 from initial conditions.

Case 1: Roots s_1, s_2 real & distinct $\left(\frac{R}{2L}\right)^2 > \frac{1}{LC}$



Exercise 4-5: $L = 1H \quad C = 10F \quad R = 4\Omega$

$$s_1, s_2 = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$= -\frac{4}{2\times 1} \pm \sqrt{\left(\frac{4}{2\times 1}\right)^2 - \frac{1}{1\times 10}}$$

$$= -2 \pm \sqrt{3\cdot 9} = -2 \pm 1.975$$

$$= -3.975, -0.025$$

$$\therefore i = A_1 e^{-3.975t} + A_2 e^{-0.025t}$$

At $t = 0^+$, $i = 0 \quad \therefore A_1 + A_2 = 0$

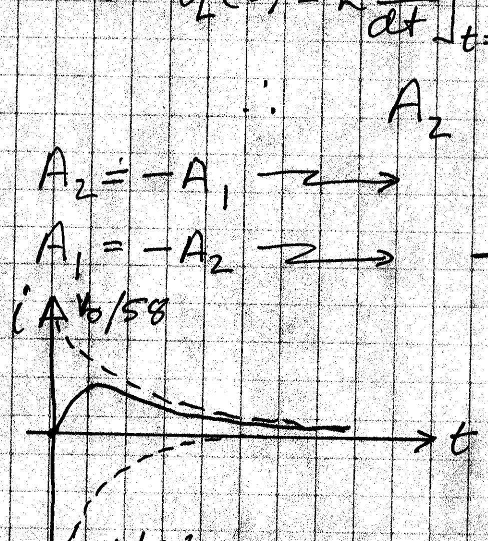
and if $i = 0$, $v_R = 0 \quad \therefore v_L = -v_C = -(-V_0)$

$$\therefore v_L(0) = L\frac{di}{dt}\Big]_{t=0} = L(-3.975A_1 - 0.025A_2) = V_0$$

$$\therefore A_2 + 59A_1 = -V_0/L$$

$A_2 = -A_1 \longrightarrow 58A_1 = -V_0/L \qquad A_1 = -V_0/58$

$A_1 = -A_2 \longrightarrow -58A_2 = -V_0/L \qquad A_2 = +V_0/58$



Roots real & distinct when $2\frac{L}{R} < \frac{RC}{2}$

— always negative
(decaying exponentials)

Overdamped system $R^2 > 4\frac{L}{C}$

Case 2: Roots s_1, s_2 complex when $R^2 < 4\frac{L}{C}$

$$s_1, s_2 = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

Write $s_1, s_2 = -\alpha \pm j\omega$ where $\alpha = R/2L$

$$\omega^2 = \frac{1}{LC} - \frac{R^2}{4L^2} = \omega_n^2 - \alpha^2$$

where $\omega_n^2 = 1/LC$

so $i = A_1 e^{(-\alpha+j\omega)t} + A_2 e^{(-\alpha-j\omega)t}$

$$= e^{-\alpha t}\left(A_1 e^{j\omega t} + A_2 e^{-j\omega t}\right)$$

$$= e^{-\alpha t}\left(A_1[\cos\omega t + j\sin\omega t] + A_2[\cos\omega t - j\sin\omega t]\right)$$

$$= e^{-\alpha t}\left[(A_1+A_2)\cos\omega t + j(A_1-A_2)\sin\omega t\right]$$

(A_1+A_2) and $j(A_1-A_2)$ must be real for real currents,

so write $A_1+A_2 = B_1$, $j(A_1-A_2) = B_2$, for

$$i = e^{-\alpha t}(B_1\cos\omega t + B_2\sin\omega t)$$

$$= Ae^{-\alpha t}\sin(\omega t+\theta)$$

Damped sinusoid

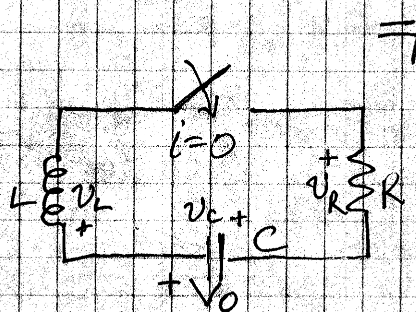
Examples 3 & 4: $L = 1H$, $C = \frac{1}{17}F$, $R = 2\Omega$

$$s_1, s_2 = -\frac{2}{2\times 1} \pm \sqrt{\left(\frac{2}{2\times 1}\right)^2 - \frac{1}{1\times 1/17}} = -1 \pm \sqrt{1-17} = -1 \pm \sqrt{-16}$$

$$= -1 \pm j4$$

$\therefore$ Response is $e^{-t}(A_1 e^{j4t} + A_2 e^{-j4t})$

$$= Ae^{-t}\sin(\omega t+\theta)$$



At $t = 0^+$ $\quad i = 0 = A\sin\theta$

$\therefore \sin\theta = 0$, $\theta = 0$ if $A \neq 0$

$L\frac{di}{dt} = V_0 \quad \therefore V_0 = Ae^{-t}\omega\cos(\omega t+\theta) - Ae^{-t}\sin(\omega t+\theta)$

$= 4A \quad \therefore A = V_0/4$

$i = \frac{V_0}{4}e^{-t}\sin 4t$ with $T = \frac{1}{f} = \frac{2\pi}{\omega} = \frac{2\pi}{4} = \frac{\pi}{2} = 1.57$ sec.

Case 3: Roots real and equal $s_1, s_2 = -\frac{R}{2L}$ when $R^2 = \frac{4}{LC}$

$= s$

$$i = A_1 e^{st} + A_2 t . e^{st}$$

Exercise 4·7 $R = 2\Omega$ $L = 0.5H$ $C = 0.5F$

(a) $s_1, s_2 = -\frac{2}{2\times .5} \pm \sqrt{\left(\frac{2}{2\times .5}\right)^2 - \frac{1}{.5\times .5}} = -2$

(b) $\therefore i = (A_1 + A_2 t)e^{-2t}$

(c) At $t = 0^+$ $i(0^+) = 0 = A_1$

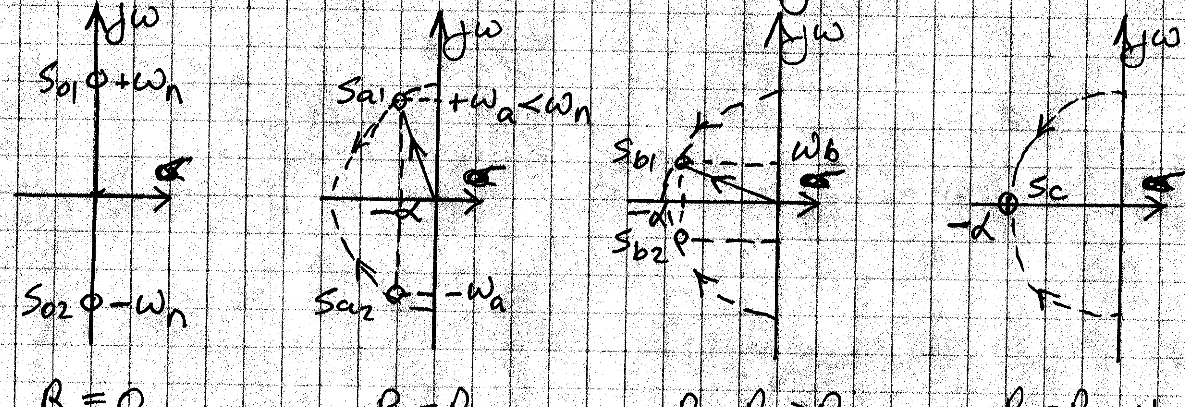
$$L\frac{di}{dt} = V_0 = L\left[A_2 t e^{-2t}(-2) + A_2 e^{-2t}\right]$$

$$\underset{t=0}{=} LA_2 \qquad \therefore A_2 = V_0/L$$

$$\therefore i = 2V_0 t e^{-2t}$$

Roots in the complex plane

$R = 0$ $\therefore s_1, s_2 = \pm\sqrt{-1/LC} = \pm j\omega_n$

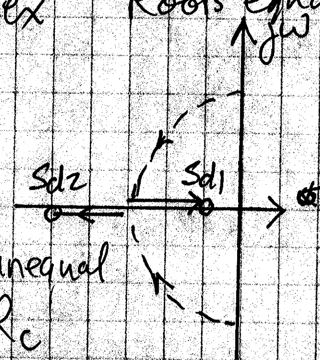


$R = 0$ Roots imag. $R = R_a$ Roots complex $R = R_b > R_a$ Roots complex $R = R_{crit}$ Roots equal

Arc radius $= (\alpha^2 + \omega^2)^{1/2}$

$= \left[\left(-\frac{R}{2L}\right)^2 + \left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)\right]^{1/2}$

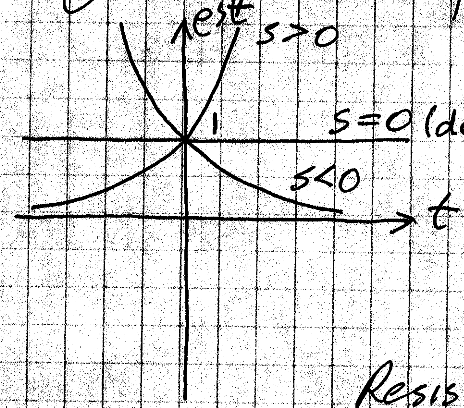
$= \frac{1}{\sqrt{LC}} = \omega_n$



Roots real, unequal

$R = R_d > R_c$

Development of generalized impedance concepts.



if $i = I_0 e^{st}$ $\quad \frac{di}{dt} = sI_0 e^{st}$
$= si$

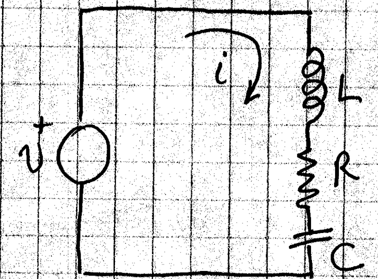
$v = V_0 e^{st}$ $\quad \frac{dv}{dt} = sV_0 e^{st}$
$= sv$

Resistance: Define $Z_R = \frac{v}{i} = \frac{iR}{i} = R$ ohms

Inductance: $v = L\frac{di}{dt}$ $\quad Z_L = \frac{v_i}{i} = \frac{sLi}{i} = sL$ ohms
$= sLi$

Capacitance: $i = C\frac{dv}{dt}$ $\quad Z_C = v/i = \frac{v}{sCv} = \frac{1}{sC}$ ohms
$= sCv$

Works for waveforms expressible as exponentials, eg. sinusoids



$v = L\frac{di}{dt} + iR + \frac{1}{C}\int i\,dt$

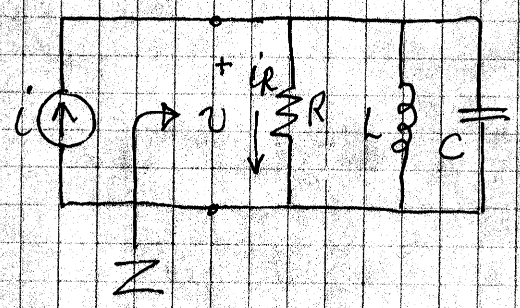
For $i = I_0 e^{st}$

$v = sLi + Ri + \frac{1}{sC}i$

& $Z = \frac{v}{i} = sL + R + \frac{1}{sC}$
$= Z_L + Z_R + Z_C$

See Examples 5, 6, & 7

Exercise 4.9



$i = e^{-5t}$ A $\quad L = 1$ H $\quad C = 1$ F $\quad R = 1\,\Omega$

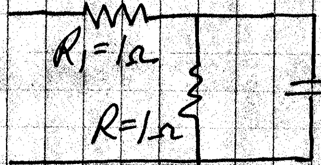
Find Z, v, i_R

$Z_R = R$ $\quad Z_L = sL = s$ $\quad Z_C = \frac{1}{sC} = \frac{1}{s}$
$= -5$ $\quad = \frac{-1}{5}$

$Z = \left(\frac{1}{R} + \frac{1}{s} + s\right)^{-1}$
$= [1 + (-.2) + (-5)]^{-1} = \frac{1}{-4.2} = -0.238\,\Omega$

$\therefore v = iZ = -0.238e^{-5t}$ V & $i_R = v/R = -0.238e^{-5t}$ A

Exercise 4.10



Plot $Z(s)$ for real s

$$Z(s) = R_1 + R \| \frac{1}{sC} = R_1 + \frac{R\frac{1}{sC}}{R + \frac{1}{sC}} = R_1 + \frac{R}{1+sRC}$$

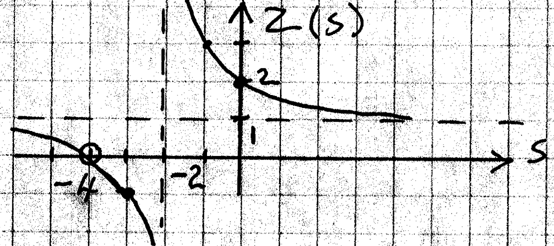
$$= \frac{R_1(1+sRC)+R}{1+sRC} = \frac{(R+R_1)+sRR_1C}{1+sRC}$$

$$= (R+R_1)\frac{1+sR_pC}{1+sRC} \quad \text{where } R_p = \frac{RR_1}{R+R_1} = R\|R_1$$
$$= 0.5\Omega$$

When $s=0$ $\quad Z(0) = R + R_1 = 2$

$Z(s) = 0 \quad 1 + sR_pC = 0 \quad s = -1/R_pC = -\frac{1}{(.5)(.5)} = -4$

As $s \to \pm\infty \quad Z(s) \to (R+R_1)\frac{sR_pC}{sRC} = \frac{(R+R_1)}{R}\frac{R_1R}{R_1+R} = R_1 = 1$



$Z(s) \to \infty$ at $s = -1/RC$
$= -2$

For $s = -1 \quad Z(-1) = 2\,\frac{1+0.25s}{1+0.5s}$
$= 2\,\frac{1-.25}{1-.5} = 3$

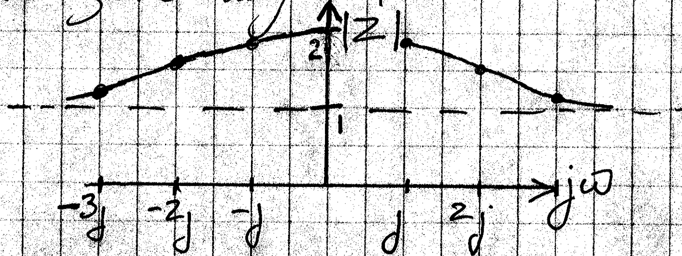
For $s = -3 \quad Z(-3) = 2\,\frac{1-.75}{1-1.5} = -1$

Exercise 4.11

(a) Plot $Z(s)$ for imaginary $s = \pm j\omega$
(b) Plot the pole-zero diagram

$$Z(s) = 2\,\frac{1+s/4}{1+s/2} = \frac{4+s}{2+s}$$

$\xrightarrow[s=0]{} 2 \qquad \xrightarrow[s\to\infty]{} 1$



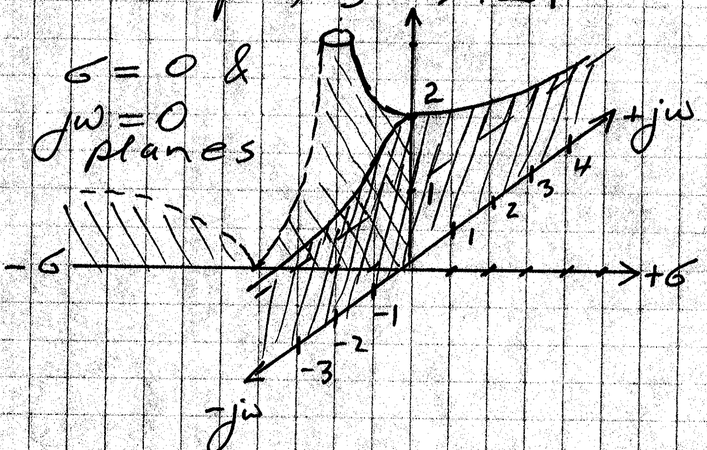
$\xrightarrow[\pm j]{} \frac{4\pm j}{2\pm j} \quad |Z| = 1.844$

$\xrightarrow[\pm 2j]{} \frac{2\pm j}{1\pm j} \quad |Z| = 1.58$

$\xrightarrow[\pm 3j]{} \frac{4\pm 3j}{2\pm 3j} \quad |Z| = 1.154$

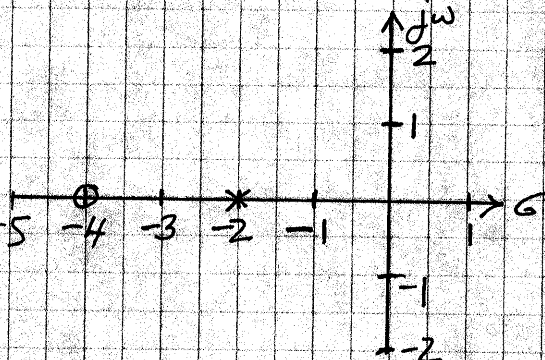
For complex s
see Fig 4.16(c) p129

Complex $s \rightarrow |Z|$



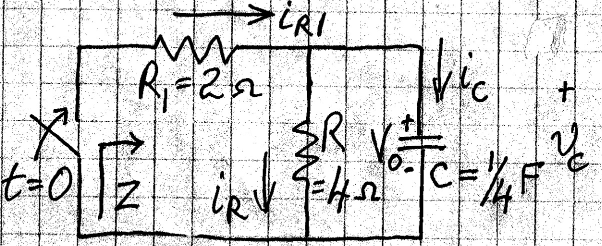
3-D surface for complex s
Hard to draw/visualize

<u>Pole-zero diagram</u>
on Complex plane.
Show where $|Z| \rightarrow \infty$
$|Z| \rightarrow 0$



In this case: Pole at $\sigma = -2$
Zero at $\sigma = -4$

Exercise 4.12. $v_c(0) = V_0$ Find $i_c(t)$ for $t > 0$
$v_c(t)$ " "

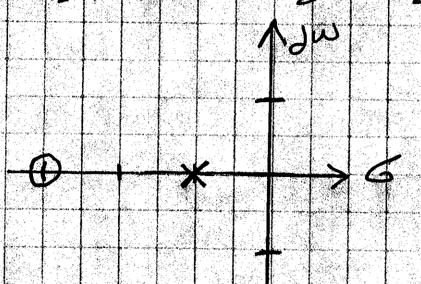


$$Z(s) = R_1 + \frac{R\,\frac{1}{sC}}{R + \frac{1}{sC}} = R_s\,\frac{1 + sR_pC}{1 + sRC}$$

where $R_s = R_1 + R = 6\,\Omega$
$R_p = R_1 \| R = \frac{4}{3}\,\Omega$

$$\therefore Z(s) = 6\,\frac{1 + s/3}{1 + s} = 2\,\frac{3 + s}{1 + s}$$

Pole when $|Z| \rightarrow \infty$
i.e. when $s = -1$
Zero when $|Z| \rightarrow 0$
i.e. when $s = -3$



Concept of a zero where $|Z| = 0$
Can have current with no forcing voltage $\rightarrow$ natural current response
(roots of characteristic equation)
$\therefore$ natural response defined by $s_1 = -3$
$i_{R1} = I_1 e^{-3t}$ & at $t = 0^+$, $I_1 = \frac{-V_0}{R_1}$

Hence $i_{R1} = -\frac{V_0}{R_1} e^{-3t}$

Current divider: $i_c = \frac{R}{R + 1/sC}\, i_{R1} = \frac{4}{4 + \frac{1}{(-3)(1/4)}}\, i_{R1} = \frac{3}{2}\, i_{R1}$

$\therefore i_c = -\frac{3}{2}\,\frac{V_0}{2}\,e^{-3t} = -0.75\,V_0\,e^{-3t}$, $v_c = -v_{R1} = -i_{R1}R_1 = +V_0\,e^{-3t}$

In general, any network impedance can be reduced to the form:

$$Z(s) = K \frac{s^n + \cdots k_2 s^2 + k_1 s + k_0}{s^m + \cdots c_2 s^2 + c_1 s + c_0}$$

$$= K \frac{(s-s_1)(s-s_2)\cdots(s-s_n)}{(s-s_a)(s-s_b)\cdots(s-s_m)}$$

where $s = s_1, s_2, \cdots s_n$ are zeros

$s = s_a, s_b, \cdots s_m$ are poles

If we have the network, $Z(s)$ can be found and poles/zeros identified.

If we have the poles & zeros, a network can be found (except K undefined)

Exercise 4.14 $\quad Z(s) = \dfrac{(s^2+4s+8)(s+1)}{s^2+2s+5}$

$$s_1, s_2 = \frac{-4 \pm (4^2 - 4\times 8)^{1/2}}{2} = -2 \pm j2$$

$$s_a, s_b = \frac{-2 \pm (2^2 - 4\times 5)^{1/2}}{2} = -1 \pm j2$$

Zeros: -1, $-2+j2$, $-2-j2$ Poles: $-1+j2$, $-1-j2$

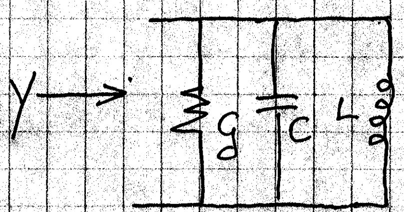
Note: All complex poles/zeros appear as conjugate pairs

Admittance: $\quad Y = \dfrac{i}{v} = \dfrac{1}{Z}$

$Y_R = \dfrac{1}{R} = G$

$Y_L = \dfrac{1}{sL}$

$Y_C = sC$



$Y(s) = G + sC + \dfrac{1}{sL}$

In general:

$$Y(s) = \frac{1}{Z(s)} = \frac{1}{K}\frac{(s-s_a)(s-s_b)\cdots(s-s_m)}{(s-s_1)(s-s_2)\cdots(s-s_n)}$$

Using poles/zeros: ① Find poles & zeros from $Z(s)$ or $Y(s)$; plot pole/zero diag.

Natural behavior:

② For SC terminals, $i = I_1 e^{s_1 t} + I_2 e^{s_2 t} + \cdots I_n e^{s_n t}$ $\quad s_1, s_2 \cdots s_n$ Z zeros, Y poles

For OC terminals, $v = V_1 e^{s_a t} + V_2 e^{s_b t} + \cdots V_m e^{s_m t}$

$s_a, s_b, \cdots s_m$ $\quad Z$ poles $\quad Y$ zeros

③ Coeffs from initial conditions