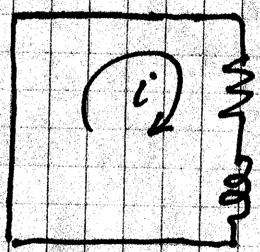
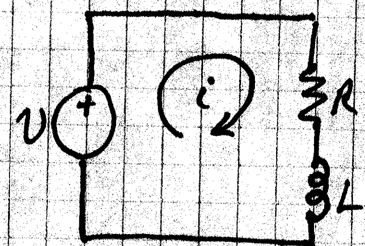
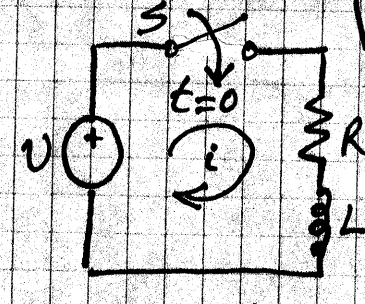
ECE 241 Lecture 10 TRANSIENTS II (Chapter 6: Complete Response)



(a) Natural response (Chapter 4)



(b) Forced response (Chapter 5)



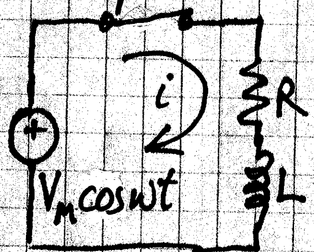
(c) Complete response (Chapter 6)

Remember (a) exponential decay from initial condition
— time constant L/R

(b) follows $e^{-t/\tau}$ for exponential forcing function.

Natural response ⟶ initially stored energy dissipates
Forced response ⟶ energy continually supplied
Complete response ⟶ combination.
Initial transient ⟶ steady state

Example:



$$L\frac{di}{dt} + Ri = V_m \cos \omega t$$

Any i which satisfies eq^n is a possible solution

e.g. $i_f = I_m(\cos \omega t + \phi)$ is the forced response (see back)

where I_m, ϕ found from $\bar{Z}$

Also, natural response: $i_n = I_n e^{-(R/L)t}$

is a solution of $L\frac{di}{dt} + Ri = 0$

So $i = i_f + i_n = I_m \cos(\omega t + \phi) + I_n e^{-(R/L)t}$

is also a solution (since i_n makes $LHS = 0$)

i_f ⟶ "particular" solution i_n ⟶ "complementary" solution

PROCEDURE

Forced response :

Same "form" as forcing function (eg. sinusoid)

Amplitude : Forcing function amplitude & Z

$Z(0)$ for DC ; $Z(j\omega)$ for sinusoid

① Write impedance $Z(s)$ or admittance $Y(s)$ function for the appropriate terminals for desired : (open circuit) voltage, or (short circuit) current

② Determine forced response from forcing function an impedance or admittance ($Z(s)$ or $Y(s)$) s defined by forcing function.

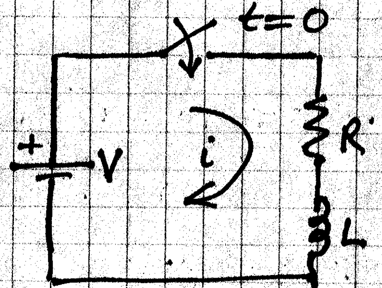
③ Identify the natural response components from the poles or zeros of $Z(s)$ or $Y(s)$

e.g. for $Z(s)$: poles $\longrightarrow$ voltage

zeros $\longrightarrow$ current

④ Add forced and natural responses, and find amplitudes from initial conditions

A. FIRST ORDER CIRCUITS



① R-L step response
(i.e. DC forcing function)

Step 1: $Z(s) = R + sL$

Step 2: Forced response $i_f = \dfrac{V}{R+sL} \xrightarrow[s=0]{DC} \dfrac{V}{R}$

Step 3: Natural components: Find current from zeros of $Z(s)$

i.e. $R+sL=0$ gives $s = -R/L$

$\therefore i_n = Ae^{-(R/L)t}$

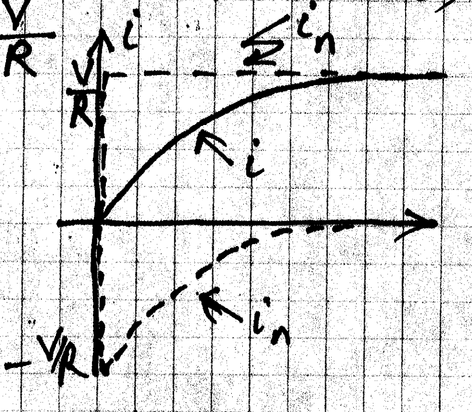
Step 4: Evaluate undetermined constants

$i = i_f + i_n = V/R + Ae^{-(R/L)t}$

At $t=0^+$ $i(0^+)=0$ (since $i(0^-)=0$ & i_L can't change instantaneously.)

$i(0) = 0 = \dfrac{V}{R} + A \quad \therefore A = -\dfrac{V}{R}$

& $i = \dfrac{V}{R}\left(1 - e^{-(R/L)t}\right)$



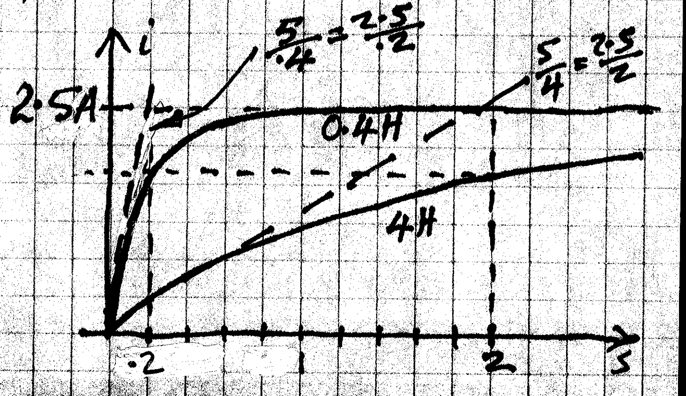
Exercise 6.1 V = 5V, R = 2Ω above. Find τ and sketch response for L = 4H, L = 0.4H

$\tau(4H) = 4H/2\Omega = 2s$
$\tau(.4H) = .4/2 = 0.2s$

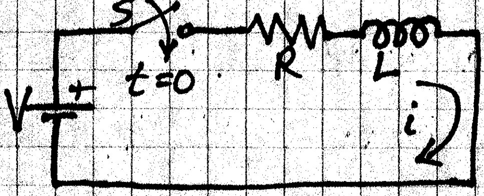
Note $L\dfrac{di}{dt} + Ri = V$

$\therefore \dfrac{di}{dt} = \dfrac{V-Ri}{L} \xrightarrow[i=0]{t=0} \dfrac{V}{L}$

$\xrightarrow[i\to V/R]{t\to\infty} 0$



Exercise 6.2 Find $i(0)$, $i(t)$, $\frac{di}{dt}$ for $t \geq 0$



This is the same circuit already solved,

so $i(t) = \frac{V}{R}\left(1 - e^{-t/\tau}\right)$

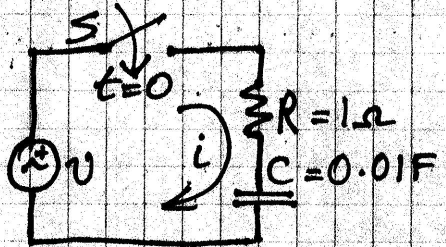
where $\tau = L/R$

$i(0) = 0$ and $\frac{di}{dt} = 0 - \frac{V}{R}\left(e^{-t/\tau}\right)\left(-\frac{1}{\tau}\right) = \frac{V}{\cancel{R}}\frac{\cancel{R}}{L}e^{-t/\tau}$

$= \frac{V}{L}e^{-t/\tau}$

② AC switching

eg. Exercise 6.3 $v = 10\sqrt{2}\cos 100t$ Find i_f, i_n



① $Z(s) = R + 1/sC$

② For sinusoidal forcing function

$\hat{Z}(j\omega) = R - j/\omega C$

$= 1 - j\frac{1}{100(.01)} = 1 - j$

$= \sqrt{2}\ \arctan -1 = \sqrt{2}\angle -45^\circ$

Hence $\vec{I} = \frac{\bar{V}}{\bar{Z}} = \frac{10\angle 0^\circ}{\sqrt{2}\angle -45^\circ}$

$= \frac{10}{\sqrt{2}}\angle 45^\circ$

$\therefore i_f = \frac{10}{\sqrt{2}}\sqrt{2}\cos(\omega t + 45^\circ)$

$= 10\cos(\omega t + 45^\circ)$ A

③ Zero of $Z(s)$ to find natural current: $R + \frac{1}{sC} = 0$

at $s = -1/RC$

$= -\frac{1}{1(.01)} = 100s$

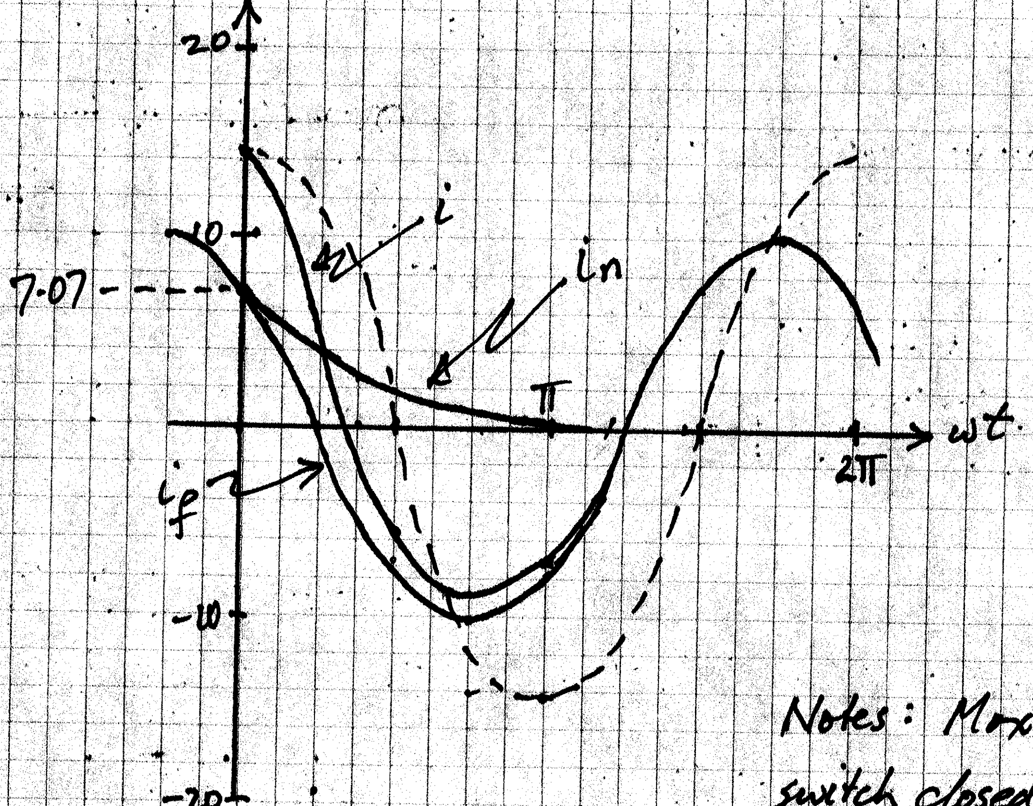
④ $i = i_n + i_f = 10\cos(\omega t + 45^\circ) + Ae^{-100t}$

At $t = 0$ $i = \frac{v(0)}{R} = \frac{10\sqrt{2}}{1} = 10\cos 45^\circ + A = \frac{10}{\sqrt{2}} + A$

$\therefore A = 10\sqrt{2} - 10/\sqrt{2} = 14.14 - 7.07$

$= 7.07$

so $i_n = 7.07e^{-100t}$ and $i = 10\cos(\omega t + 45^\circ) + 7.07e^{-100t}$



$\omega t = 100t = 2\pi$ when $t = 62.84$ ms

$\omega t = \pi$ at $t = 31.84$ ms

$\tau = 0.01$ s $= 10$ ms.

Notes: Max transient current if switch closed at max v_{IN}

Also, no transient i_n if $A=0$, i.e. if switch closed at $t=T$ such that $10\sqrt{2}\cos 100T = 10\cos(100T + 45^\circ)$

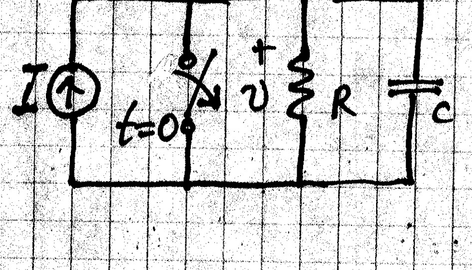
$$\cos 100T = \frac{1}{\sqrt{2}}(\cos 100T \cos 45^\circ - \sin 100T \sin 45^\circ)$$

$$= (\cos 100T - \sin 100T)/2$$

i.e. when $\sin 100T = -\cos 100T$

$$\tan 100T = -1, \quad 100T = -45^\circ$$

③ Parallel Circuits



Employ duality or by inspection plus experience —

$v_f = IR$

$v = IR(1 - e^{-t/RC})$

$\therefore v_n = -IRe^{-t/RC}$

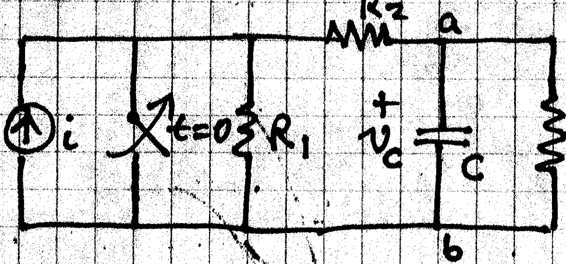
$Y(s) = \frac{1}{R} + sC$ $\quad v_f$ for dc, $Y(0) = 1/R$: $v_f = I/Y = IR$

Zeros of $Y(s)$ to find voltage → $s = -1/RC$ $\therefore v_n = Ae^{-t/RC}$

At $t = 0^+$ $IR + Ae^{-t/RC} = 0 = IR + A$ $\therefore A = -IR$ & $v_n = -IRe^{-t/RC}$

$\therefore v = IR(1 - e^{-t/RC})$

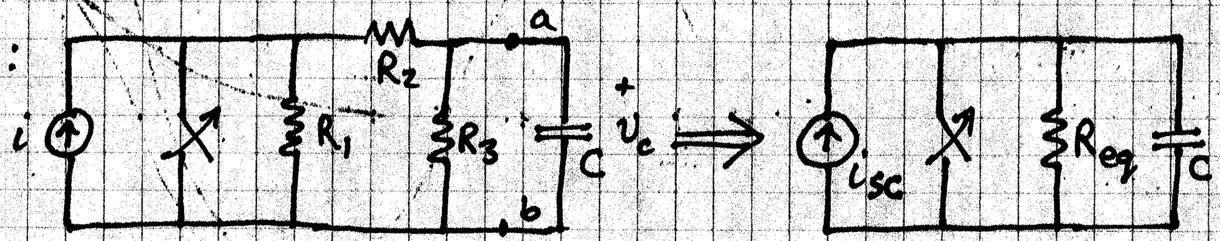
Exercise 6.4



For $v_C(t)$:

Find natural response τ

Redraw:



$\tau = R_{eq}C$ where $R_{eq} = R_3 \,\|\, (R_1+R_2)$ $\therefore$ $\tau = \dfrac{R_3(R_1+R_2)C}{R_1+R_2+R_3}$

&

$i_{sc} = \dfrac{R_1}{R_1+R_2}\, i$

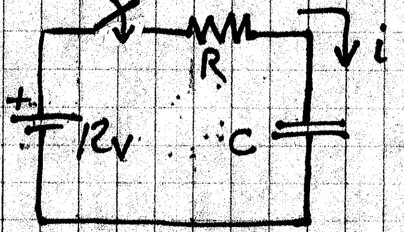
Hence $v_C(t) =$

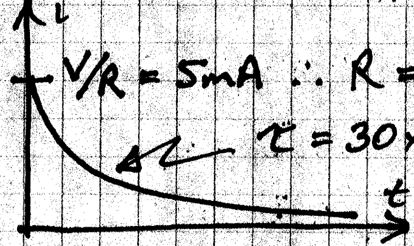
$$\frac{R_1}{R_1+R_2}\, i\, R_{eq}\left(1-\exp -t/\tau\right)$$

$$= \frac{R_1R_3\, i}{R_1+R_2+R_3}\left(1-\exp -t/\tau\right)$$

Exercise 6.5:

Given 12v battery, design circuit for current in one element to jump to 5mA at $t=0^+$ when switch closed, then decay exponentially with $\tau = 30$ms.





$V/R = 5$mA $\therefore$ $R = \dfrac{12\text{v}}{5\times10^{-3}} = 2.4\text{k}\Omega$

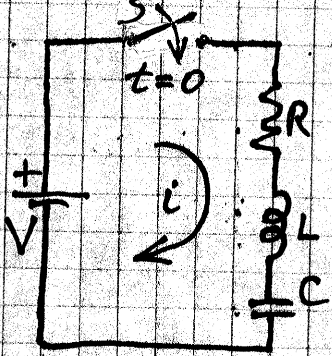
$\tau = 30\times10^{-3} = RC = 2.4\times10^3 C$

$\therefore C = \dfrac{30\times10^{-3}}{2.4\times10^3}$

$= 12.5\mu F$

B. SECOND ORDER CIRCUITS

① Step Response



① $Z(s) = R + sL + \frac{1}{sC}$

② Forced response

$$i_f = \frac{V}{Z(0)} \quad \text{since forcing function is DC } s=0$$
$$= \frac{V}{\infty} = 0$$

③ Natural response components from $Z(s)=0$

$$s^2L + sR + \frac{1}{C} = 0$$

has roots $s_1, s_2 = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} = -\alpha \pm j\omega$

For roots real & distinct $\left(R^2 > \frac{4L}{C}\right)$ $\quad i_n = A_1 e^{s_1 t} + A_2 e^{s_2 t}$

For roots real & equal $\left(R^2 = \frac{4L}{C}\right)$ $\quad i_n = A_1 e^{st} + A_2 t e^{st}$

For roots complex $\left(R^2 < \frac{4L}{C}\right)$ $\quad i_n = e^{-\alpha t}(B_1 \cos\omega t + B_2 \sin\omega t)$
$$= A e^{-\alpha t} \sin(\omega t + \theta)$$

④ In all cases, 2 constants need to be determined from initial cond.'s

For $i(0) = 0 = i_n + i_f$ assuming as here $i(0)=0$

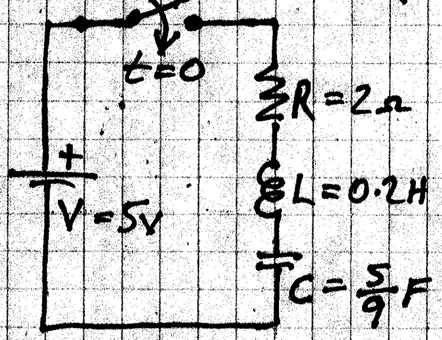
& assuming here $v_C(0) = 0$ and $v_R(0) = iR = 0$

then $L \left.\frac{di}{dt}\right]_{t=0} = V$ ie. $\frac{di}{dt} = \frac{V}{L}$

ie. 2 conditions from which to find the two constants.

(The details will vary for different cases.)

Exercise 6.6 (See also Example 4)



Find i, v_L, $\frac{di}{dt}$ at $t=0^+$

$i(0^+) = 0$

$v_L(0^+) = 5V$ since $v_C(0) = 0$ & $Ri(0) = 0$

$$\left.\frac{di}{dt}\right]_{t=0^+} = \frac{v_L(0^+)}{L} = \frac{5V}{0.2H} = 25/s$$

From previous slide and for $R^2 = 4 > \frac{4L}{C} = \frac{4\times(0.2)}{5/9} = 1.44$

i.e. real, distinct roots

$$i = 0 + A_1 e^{s_1 t} + A_2 e^{s_2 t} \qquad i(0) = A_1 + A_2 = 0$$

$$A_2 = -A_1$$

$$\left.\frac{di}{dt}\right]_{t=0} = s_1 A_1 + s_2 A_2 = 25$$

$$s_1, s_2 = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} = -\frac{2}{.4} \pm \sqrt{\left(\frac{2}{.4}\right)^2 - \frac{1}{.2(5/9)}}$$

$$= -5 \pm \sqrt{25-9} = -5 \pm \sqrt{16} = -5 \pm 4$$

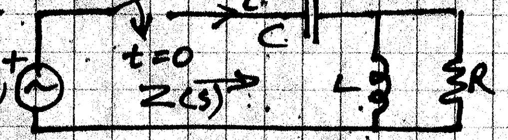
$$= -9, -1$$

$\therefore -9A_1 - A_2 = 25 \rightarrow -9A_1 + A_1 = 25$, $A_1 = -25/8$ $\quad A_2 = +25/8$

$$i(t) = \frac{25}{8}\left(e^{-t} - e^{-9t}\right)$$

② AC Switching Transients (See Example 5 + Exercise 6.7)

Exercise 6.8 $R = 1\Omega$ $C = 1F$ $L = 1H$ $v = \sqrt{20}\cos(t - 149^\circ)$ V



(a) Find $Z(s)$, $i_f(t)$

(b) Find roots of $Z(s)$ and complete response

Step ① $$Z(s) = \frac{1}{sC} + \frac{R \cdot sL}{R + sL} = \frac{(R + sL) + sRL \cdot sC}{sC(R + sL)}$$

$$= \frac{s^2 RLC + sL + R}{sC(R + sL)} = \frac{s^2 + s\frac{1}{RC} + \frac{1}{LC}}{s(s + R/L)}$$

$$= (s^2 + s + 1)/s(s+1)$$

Step ② Forcing function $i_f = \dfrac{V}{Z(s)}$ where $s = j\omega$ for sinusoidal input v

$$= \frac{\sqrt{10}\angle -149^\circ}{Z(j\omega)} \text{ and } \omega = 1$$

$$Z(j\omega) = \frac{-\omega^2 + j\omega}{j\omega(j\omega + 1)} = \frac{-1^2 + 1 + j}{j(1 + j)} = \frac{j}{-1 + j} = j\frac{(-1 - j)}{1 + 1}$$

$$= \tfrac{1}{2}(1 - j) = \frac{1}{\sqrt{2}}\angle -45^\circ \qquad \therefore i_f = \frac{\sqrt{10}\angle -149^\circ}{1/\sqrt{2}\angle -45^\circ} = \sqrt{20}\angle -104^\circ$$

$$\therefore i_f = \sqrt{2}\sqrt{20}\cos(t - 104^\circ) = 2\sqrt{10}\cos(t - 104^\circ)$$ (Book answer incorrect)

Step ③ Zeros of $Z(s)$ for current: $Z(s) = s^2 + s + 1$

$$s_1, s_2 = -\frac{1}{2} \pm \sqrt{(1/2)^2 - 1} = -\frac{1}{2} \pm j\sqrt{3/4}$$

$$= -\frac{1}{2}(1 \pm j\sqrt{3}) = -\alpha \pm j\omega$$

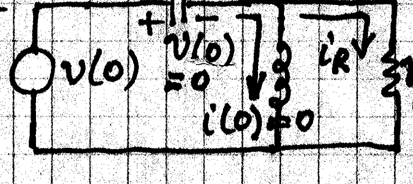
$$\therefore i_n = A e^{-t/2} \sin\left(\frac{\sqrt{3}}{2}t + \theta\right)$$

Step ④ Initial conditions $i(0) = 2\sqrt{10}\cos(-104^\circ) + A\sin\theta$

$$= \frac{v(0)}{R} = \sqrt{20}\cos(-149^\circ)$$

$$\therefore A\sin\theta = \sqrt{20}\cos(-149^\circ) - 2\sqrt{10}\cos(-104^\circ) = -5.36$$

Also at $t = 0^+$



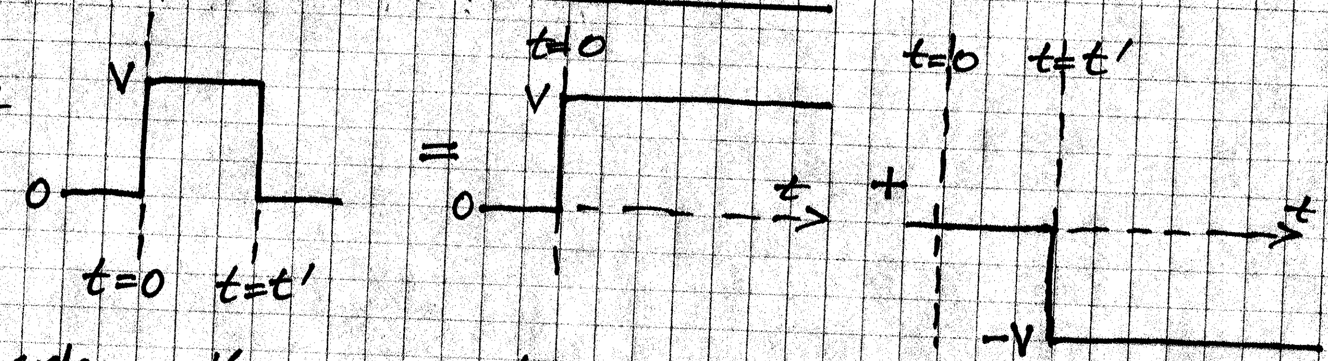
$$v(0) = L\left.\frac{di_L}{dt}\right]_{t=0} = L\left.\frac{di}{dt}\right]_{t=0}$$

$$\left.\frac{di}{dt}\right]_{t=0} = -2\sqrt{10}\sin(-104^\circ) + A\left[\cos\theta + \sin\theta(-\tfrac{1}{2})\right]$$

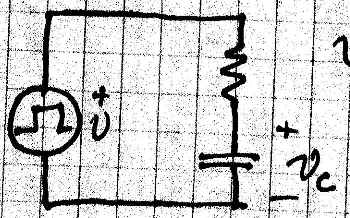
$$\therefore A\cos\theta = 2\sqrt{10}\left[\cos(-104^\circ) + \sin(-104^\circ)\right] + \tfrac{1}{2}A\sin\theta = -10.345$$ Hence A, θ.

PULSE / IMPULSE RESPONSES

Pulse



Consider as the response to 2 steps

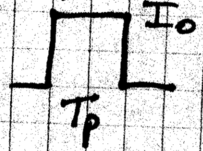


$v_C = V(1 - e^{-t/RC})$ $\quad 0 < t < t'$

and $v_C = V(1 - e^{-t/RC}) - V(1 - e^{-(t-t')/RC})$ $\quad t' < t$

Impulse:

Current pulse $I_0 T_p = Q_0$



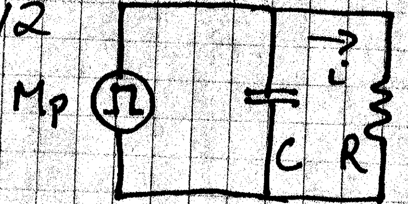
Impulse $\left.\begin{array}{l} T_p \to 0 \\ I_0 \to \infty \end{array}\right\} I_0 T_p = Q_0$

$M_p = \int i\,dt$ for arbitrary shape
$= CV$

where M_p is impulse "magnitude."

Similarly: $M_p = \int v\,dt = LI$

Exercise 6.12



M_p A.s at $t=0$

Find M_p to establish current $i_R(t) = I_0 e^{-\alpha t}$

Impulse charges capacitor with infinite current zero time

$$M_p = CV_0 = C(I_0 R)$$