Density of States

- Need to know the density of electrons, \( n \), and holes, \( p \), per unit volume
- To do this, we need to find the density of permitted energy states and then the probability (given by the Fermi function) that these states are occupied by an electron
- See Neamen Sections 3.4 and 3.5 for derivations, and compare these with alternative approaches given below
Density of States

- See figure in “k-space.”
- Also: $p=\hbar k=(\hbar/2\pi)k$, i.e.
  - Direct linear relationship between (wave-vector) $k$-space and (momentum) $p$-space.
- The Neamen derivation (next slide) is based in $k$-space.
- See also a slightly different approach based in $p$-space in following slide.

Consider electron confined to crystal (infinite potential well) of dimensions $a$ (volume $V=a^3$)
- It has been shown that $k=n\pi/a$, so $\Delta k=k_{n+1}-k_n=\pi/a$
- Each quantum state occupies volume $(\pi/a)^3$ in $k$-space.
- Number of quantum states in range $k$ to $k+dk$ is $4\pi k^2 \cdot dk$
  and the number of electrons in this range $k$ to $k+dk$ is
  
  $g_T(k)dk = 2(1/8) \cdot 4\pi k^2 \cdot dk / (\pi/a)^3 = (k^2 \cdot dk / \pi^2) a^3$

  where “2” comes from spin degeneracy, and “1/8” because $E=(\hbar k)^2/2m$,
  so negative $k$ values do not add more energy states,
  i.e. use $k_x, k_y, k_z > 0$ quadrant only
- Converting $k$-space to energy ………..
Density of States

\[ E = E_c + \frac{p^2}{2m^*_e} \] in the conduction band, where \( E > E_c \)

\[ \therefore \frac{p^2}{\hbar^2} = k^2 = \frac{2m^*_e}{\hbar^2} (E - E_c) \]

\[ 2k \cdot dk = 2m^*_e \frac{dE}{\hbar^2} \]

\[ \therefore \frac{k^2 \cdot dk}{\hbar^2} = \frac{(2m^*_e)^{\frac{1}{2}}}{\hbar^2} (E - E_c)^{\frac{1}{2}} m^*_e \frac{dE}{\hbar^2} \]

So writing \( g_e(k)dk/V = g_c(E)dE = \left( \frac{k}{\pi} \right)^2 \frac{dk \cdot a^2}{a^3} = \frac{1}{\pi} \left( \frac{2m^*_e}{\hbar^2} \right)^{\frac{1}{2}} (E - E_c)^{\frac{1}{2}} m^*_e \frac{dE}{\hbar^2} \)

\[ = \frac{4\pi}{\hbar^3} \left( \frac{2m^*_e}{\hbar^2} \right)^{\frac{3}{2}} (E - E_c)^{\frac{3}{2}} dE = 4\pi \left( \frac{2m^*_e}{\hbar^2} \right)^{\frac{3}{2}} (E - E_c)^{\frac{1}{2}} dE \]

is electron density of states/unit energy/unit vol in the conduction band
and electron density/unit energy/unit vol in the conduction band is

\[ n(E)dE = f_{\text{Ti}}(E)g_c(E)dE \]

Density of States (alternate)

Heisenberg says \( \Delta p \Delta x = \hbar \),

(but see \( \hbar \) in Neamen section 2.1.3 p30, and \( \hbar/2 \) in Streetman)
and for crystal volume \( V = L_xL_yL_z \)

\[ \Delta p_x \Delta x = \hbar, \Delta p_y \Delta y = \hbar, \Delta p_z \Delta z = \hbar, \]
and hence for a single electron

\[ \Delta p_x \Delta p_y \Delta p_z = \frac{\hbar}{V} \]
for \( \Delta x = L_x \), etc
is the “volume” occupied in p-space by one electron.

Number of allowed conduction electron states \( g(p)dp \) from \( p \) to \( p+dp \)
[see Fig 3.26 (slide 3) in \( k \)-space with \( p = \hbar k \)]

is the volume of the p-space “shell” containing all states from \( p \) to \( p+dp \)

\[ 4\pi p^2 dp \]
divided by the p-space volume of each electron, i.e. ……
Density of states

\[
g(p)dp = 2 \frac{\left(4\pi p^2 dp\right)}{\hbar/V} = \frac{V}{\hbar} \frac{2(4\pi p^2 dp)}{\hbar} \text{ where the 2 is due to spin degeneracy}
\]

To convert to energy density:

\[
E = E_c + \frac{p^2}{2m^*} \text{ in the conduction band, where } E > E_c
\]

\[
: p^2 = 2m^*(E - E_c)
\]

\[
2p dp = 2m^* dE
\]

\[
: p^2 dp = \left(2m^*\right)^{1/2}(E - E_c)^{1/2} m^*_d dE
\]

So writing \(g(p)dp/V = g_c(E)dE = \frac{2m^*}{\hbar^3} 4\pi(2m^*)^{1/2}(E - E_c)^{1/2} dE\)

\[
= \frac{4\pi}{\hbar^3} (2m^*)^{3/2}(E - E_c)^{1/2} dE = 4\pi(2m^*)^{1/2}(E - E_c)^{1/2} dE
\]

is electron density of states/unit energy/unit vol in the conduction band
and electron density/unit energy/unit vol in the conduction band is

\[
m(E)dE = f_{E_0}(E)g_c(E)dE
\]

(Compare 1D, 2D systems with 3D)

(Compare 1D, 2D systems with 3D)

\[
g(k)dk = 2 \frac{\left(\frac{1}{8}4\pi k^2 dk\right)}{\pi a^2} \left(\frac{\pi}{a}\right)^3 \text{ in 3D}
\]

\[
= 2 \frac{\left(\frac{1}{4}\right)2\pi k^2 dk\left(\frac{\pi}{a}\right)^2}{\pi a^2} \text{ in 2D}
\]

\[
= 2 \left(\frac{1}{2}\right) d\left(\frac{\pi}{a}\right) \text{ in 1D}
\]

And similarly for holes:

For electrons in the conduction band, where $E > E_C$:

$$g_C(E) = \frac{4\pi}{h^3} \left(2m^*_e\right)^{\frac{3}{2}} \sqrt{E - E_C}$$

For holes in the valence band, where $E < E_V$:

$$g_V(E) = \frac{4\pi}{h^3} \left(2m^*_p\right)^{\frac{3}{2}} \sqrt{E - E_V}$$

and $g(E) = 0$ for $E_V < E < E_C$

Now that we have the densities of states, we need the probabilities that these are filled.

Ex 3.3 Calculate the density of quantum states ($/cm^3$) for a free electron over the energy range of (a) $0 \leq E \leq 2eV$ & (b) $1 \leq E \leq 2eV$

(a) $N = \frac{4\pi(2m)^{3/2}}{h^3} \int_0^{2eV} \sqrt{E} \cdot dE$

$$= \frac{4\pi(2m)^{3/2}}{h^3} \cdot \frac{2}{3} \cdot E^{3/2} \bigg|_0^{2eV}$$

$$= \frac{4\pi}{h^3} \left[\frac{(9.11 \times 10^{-31})^{1/2}}{6.625 \times 10^{-34}}\right]^{3/2} \times \frac{2}{3} \left[2(1.6 \times 10^{-19})\right]^{1/2}$$

$$= 1.28 \times 10^{22} \text{ m}^{-3} = 1.28 \times 10^{22} \text{ cm}^{-3}$$

(b) $N = \frac{4\pi(2m)^{3/2}}{h^3} \cdot \frac{2}{3} \cdot E^{3/2} \bigg|_{1eV}^{2eV}$

$$= \frac{1.06286 \times 10^{27} \times \frac{2}{3} \left[2(3^{3/2} - 1^{3/2})(1.6 \times 10^{-19})\right]^{1/2}}{h^3}$$

$$= 8.29 \times 10^{27} \text{ m}^{-3} = 8.29 \times 10^{21} \text{ cm}^{-3}$$
Ex 3.3 Calculate the density of quantum states (l/cm³) in Si from (Eᵥ-kT) to Eᵥ at T=300 K

\[ N = \int_{E_{v} - kT}^{E_{v}} \frac{4\pi(2m_{p}^{*})^{3/2}}{\hbar^{3}} \sqrt{E_{v} - E} \cdot dE \]

\[ = \frac{4\pi(2m_{p}^{*})^{3/2}}{\hbar^{3}} \cdot \left( \frac{-2}{3} \right) \cdot (E_{v} - E)^{3/2} \bigg|_{E_{v} - kT}^{E_{v}} \]

\[ = \frac{4\pi \left[ (0.56)(9.11 \times 10^{-21}) \right]^{3/2}}{\left( 6.625 \times 10^{-34} \right)^{3}} \times \left( \frac{-2}{3} \right) \times (0.0259)(1.6 \times 10^{-19})^{3/2} \]

\[ = 7.92 \times 10^{24} \text{ m}^{-3} = 7.92 \times 10^{18} \text{ cm}^{-3} \]

Fermi-Dirac statistics (see also text “derivation”)

- First, let’s find the Maxwell-Boltzmann distribution
- Consider a system of identical particles with random collisions
- Assume 2 particles of initial energies E₁ & E₂ collide with energy transfer δ to give final energies E₃ & E₄
- Conservation of Energy requires that: \( E₁ + E₂ = E₃ + E₄ = (E₁ + δ) + (E₂ - δ) \)
- If \( p(E) \)=probability that particle has energy \( E \), then:

  Probability of collision = \( C \cdot p(E₁) \cdot p(E₂) \)

  Equilibrium: equal numbers of forward & reverse collisions for constant energy distribution, i.e.

  \[ C \cdot p(E₁) \cdot p(E₂) = C \cdot p(E₁) \cdot p(E₄) \]
  \[ p(E₁) \cdot p(E₂) = p(E₁ + δ) \cdot p(E₂ - δ) \]
  \[ \frac{p(E₁ + δ)}{p(E₁)} = \frac{p(E₂ - δ)}{p(E₂)} = \lambda \text{ (constant)} \]
  \[ \text{i.e.} \ p(E₁ + δ) = λ \cdot p(E₁) \]
**MB continued & FD intro**

Result \( p(E_i + \delta) = \lambda \cdot p(E_i) \) suggests an exponential distribution, i.e.

\[
p_{MB}(E) = A \cdot \exp(-\alpha E)
\]

\[
= A \cdot \exp(-E/kT)
\]

- Check: \( A \cdot \exp(-\alpha (E_i + \delta)) = \lambda \). \( A \cdot \exp(-\alpha E_i) \) if \( \lambda = \exp(-\alpha \delta) \)

- For Fermi-Dirac statistics, repeat as for MB, but with the Pauli Exclusion Principle included, i.e.

- Can only get transfer of energy \( E_1, E_2 \rightarrow E_3, E_4 \)

  if \( E_1 \) & \( E_2 \) are occupied AND if \( E_3 \) & \( E_4 \) are unoccupied

---

**Fermi function**

So the collision probability is \( C \cdot p(E_1) \cdot p(E_2) \cdot [1 - p(E_1)] \cdot [1 - p(E_2)] \)

and applying Equilibrium and Energy Conservation again:

\[
C \cdot p(E_{1i}) \cdot p(E_{2i}) \cdot [1 - p(E_{1i} + \delta)] \cdot [1 - p(E_{2i} - \delta)] = \frac{1 - p(E_{1i})}{p(E_{1i})} \cdot \frac{p(E_{1i} + \delta)}{1 - p(E_{1i} + \delta)} = \frac{1 - p(E_{2i} - \delta)}{p(E_{2i} - \delta)} \cdot \frac{p(E_{2i})}{1 - p(E_{2i})}
\]

i.e.

\[
1 - p(E_{1i}) = A \cdot \exp(-\beta E) \text{ (where } \beta = A \cdot \exp(2\beta E)\text{)}
\]

i.e.

\[
p(E) = \frac{1}{1 + A \cdot \exp(-\beta E)} = \frac{1}{1 + A \cdot \exp(-\beta E)} \text{ for } \beta \text{ large,}
\]

\[
\beta = \frac{1}{kT}
\]

and writing \( \frac{1}{A} = \exp\frac{E_f}{kT}, \ f_{FD} = \frac{1}{1 + \exp\frac{E - E_f}{kT}} \)

---
MB/FD/(BE) comparisons

<table>
<thead>
<tr>
<th>Maxwell-Boltzmann</th>
<th>Bose-Einstein</th>
<th>Fermi-Dirac</th>
</tr>
</thead>
<tbody>
<tr>
<td>Particles</td>
<td>Particles indistinguishable (wave functions overlap – cannot find individual particle)</td>
<td></td>
</tr>
<tr>
<td>distinguishable</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Any number of particles can be in the same energy state</td>
<td>Only one particle in any energy state (Pauli) but note spin degeneracy</td>
<td></td>
</tr>
<tr>
<td>Particles are statistically independent</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sum_{i=1}^{M} N_i = N$</td>
<td>$\sum_{i=1}^{N} E_i = E$</td>
<td></td>
</tr>
<tr>
<td>Atoms, molecules, etc</td>
<td>Bosons</td>
<td>Fermions</td>
</tr>
<tr>
<td></td>
<td>Photons, phonons, $^4$He nuclei (integral spins)</td>
<td>Electrons, protons, neutrons, $^3$He nuclei (½-odd-integral spins)</td>
</tr>
</tbody>
</table>

Fermi distributions

- Electrons in solids obey Fermi-Dirac statistics due to the Pauli exclusion principle (each state can have only one electron – but remember spin!)
- Probability density function gives the ratio of filled to total allowed states at a given energy.
- Using statistical mechanics to count states we find the Fermi-Dirac distribution function:
  - $f(E) = \left\{ 1 + \exp\left(\frac{E-E_F}{kT}\right) \right\}^{-1}$
  - $k$ is Boltzmann’s constant $= 8.62 \times 10^{-5}$ eV/K $= 1.38 \times 10^{-23}$ J/K
  - $E_F$ is the Fermi level, [where $f(E_F)=\frac{1}{2}$]
The Fermi-Dirac distribution function.

\[ T = 0^\circ K : \quad E > E_F \rightarrow \frac{1}{1 + \exp \frac{E - E_F}{kT}} = 0 \]

\[ E < E_F \rightarrow \frac{1}{1 + \exp \frac{E - E_F}{kT}} = 1 \]

For all \( T \)

\[ f(E_F) = \frac{1}{2} \]

At \( T = 0 \) K

Figure 3.29 | The Fermi probability function versus energy for \( T = 0 \) K.

Figure 3.30 | Discrete energy states and quantum states for a particular system at \( T = 0 \) K.

Figure 3.31 | Density of quantum states and electrons in a continuous energy system at \( T = 0 \) K.
At $T > 0\ K$

- The Fermi distribution gives the probability that an available state will be occupied.
  - Within the band gap there are no available states.
  - Even with non-zero $f(E)$ there is no occupancy in the gap. (An energy state at the Fermi level would have a probability of ½ of being occupied, if there was one.)
- At 0K every available state up to the Fermi level is occupied and every state above the Fermi level is empty.
- Fermi function is "symmetric" about the Fermi level
- For holes:

$$1 - f(E) = 1 - \frac{1}{1 + \exp \frac{E - E_F}{kT}} = \frac{\exp \frac{E - E_F}{kT}}{1 + \exp \frac{E - E_F}{kT}} = \frac{1}{1 + \exp \frac{E_F - E}{kT}}$$
Fermi function symmetry: electrons and holes

Figure 3.34 | The probability of a state being occupied, \( f(E) \), and the probability of a state being empty, \( 1 - f(E) \).

Approximate FD → MB for \((E-E_F)>>kT\)

See Example 3.6: 
\[ f(E_F+3kT) \approx 5\% \text{ for } 300K \]
So \( f(E) \) varies quickly

Example 3.8:
\[
\frac{(f_{\text{MB}}-f_{\text{FD}})}{f_{\text{FD}}} = 5\% \\
\text{at } E-E_F = kT \ln(1/0.05) \approx 3kT
\]

Note \( f(E) = \left\{ 1 + \exp[(E-E_f)/kT] \right\}^{-1} \)
\[ \approx \exp -[(E-E_f)/kT] \]
for 
\[ (E-E_F)>>kT^* \text{ condition} \]
where 
\[ \exp[(E-E_f)/kT] >> 1 \]
Ex 3.6 Assume $E_F = E_C - 0.3\, \text{eV}$ and $T = 300\, \text{K}$. Determine probability of state occupied at (a) $E = E_C + kT/4$ & (b) $E = E_C + kT$

(a) $f_x(E) = \exp\left[-\frac{(E - E_F)}{kT}\right]$

$= \exp\left[-\frac{(E_C + kT/4 - E_F)}{kT}\right]$

$= \exp\left[-\frac{(0.30 + 0.0259/4)}{0.0259}\right]$

$= 7.26 \times 10^{-6}$

(b) $f_x(E) = \exp\left[-\frac{(E - E_F)}{kT}\right]$

$= \exp\left[-\frac{(0.30 + 0.0259)}{0.0259}\right]$

$= 3.43 \times 10^{-4}$

Ex 3.7 Assume $E_F = E_C - 0.3\, \text{eV}$. Determine $T$ at which the probability of state at $E = E_C + 0.025\, \text{eV}$ is occupied is $8 \times 10^{-6}$.

$$ f_x(E) = \exp\left[-\frac{(E - E_F)}{kT}\right] $$

$8 \times 10^{-6} = \exp\left[-\frac{(0.30 + 0.025)}{kT}\right]$,

$\exp\left[\frac{-0.325}{kT}\right] = 1.25 \times 10^5$

$\frac{0.325}{kT} = \ln(1.25 \times 10^5) = 11.736$

$kT = \frac{0.325}{11.736} = 0.02769 = (0.0259)\frac{T}{300}$

$T = 321\, \text{K}$
Ex. 3.8 Calculate the energy, in terms of $kT$ and $E_F$, at which the difference between the FD and MB distributions is 2% of the FD value.

\[
\exp\left[-\frac{(E - E_F)}{kT}\right] \cdot \frac{1}{1 + \exp\left[\frac{E - E_F}{kT}\right]} = 0.02
\]

\[
\exp\left[-\frac{(E - E_F)}{kT}\right] \cdot \frac{1}{1 + \exp\left[\frac{E - E_F}{kT}\right]} - 1 = 0.02
\]

Then

\[
\exp\left[-\frac{(E - E_F)}{kT}\right] = 0.02
\]

\[
E - E_F = kT \ln\left(\frac{1}{0.02}\right) = 3.9kT
\]

Electron & hole densities

\[n(E) = g_c(E).f(E)\]

\[p(E) = g_v(E).[1-f(E)]\]

Figure 4.1 (a) Density of states functions, Fermi-Dirac probability function, and areas representing electron and hole concentrations for the case when $E_F$ is near the midgap energy; (b) expanded view near the conduction-band energy; and (c) expanded view near the valence-band energy.

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Effective density of states & Fermi integral

Electron density in the conduction band

\[ n_n = \int_{E_c}^{E_F} n(E)dE = \int_{E_c}^{E_F} f_{S1}(E)g_c(E)dE \]

where \( E_c \) is the top of the conduction band

\[ \approx \frac{2m^*}{\hbar^2} \int_{E_c}^{E_F} \left( \frac{E - E_c}{kT} \right)^{1/2} \] 

\[ \times \left[ 1 + \exp \left( \frac{E - E_c}{kT} \right) \right]^{-1} dE \]

\[ = \frac{4\pi}{\hbar^2} \frac{2m^*}{\hbar^2} F_{\eta_F}(\eta) \]

where \( F_{\eta_F}(\eta) = \int_0^1 \frac{\eta^2 d\eta}{1 + \exp(\frac{\eta - \eta_s}{kT})} \), \( \eta = \frac{E - E_c}{kT}, \eta_s = \frac{E_F - E_c}{kT} \)

This is the (tabulated) Fermi Integral

A similar treatment gives a similar result for \( g_v(E) \) and \( p_p \) for holes in the valence band

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Effective density of states

For conduction electrons with \( E - E_F \gg kT \) (usually so)

\[ f_{S1b}(E) = f_{S1}(E) = \exp \left( -\frac{E - E_F}{kT} \right) \]

so \( n_n \approx \frac{2m^*}{\hbar^2} \int_{E_c}^{E_F} \left( \frac{E - E_c}{kT} \right)^{1/2} \exp \left( -\frac{E - E_F}{kT} \right) dE = \frac{2m^*}{\hbar^2} \int_{E_c}^{E_F} \left( \frac{E - E_c}{kT} \right)^{1/2} \exp \left( -\frac{E - E_F}{kT} \right) \] 

\[ \times \left[ 1 + \exp \left( \frac{E - E_c}{kT} \right) \right]^{-1} dE \]

[writing \( x = \frac{E - E_c}{kT}, dE = kTdx, \exp \left( -\frac{E - E_F}{kT} \right) = \exp \left( -\frac{E_c - E_F}{kT} \right) \exp \left( -\frac{E - E_c}{kT} \right) \), and using \( \int_0^1 x e^{-x}dx = \frac{1}{2} \)]

\[ = \frac{4\pi}{\hbar^2} \frac{2m^*kT}{\hbar^2} \exp \left( -\frac{E_c - E_F}{kT} \right) \frac{\pi^{1/2}}{2} \times \left[ 2 \left( \frac{2m^*kT}{\hbar^2} \right)^{1/2} \exp \left( -\frac{E_c - E_F}{kT} \right) \exp \left( -\frac{E - E_c}{kT} \right) \right] \]

\[ = 4\pi \frac{2m^*kT}{\hbar^2} \exp \left( -\frac{E_c - E_F}{kT} \right) \frac{\pi^{1/2}}{2} \times \left[ 2 \left( \frac{2m^*kT}{\hbar^2} \right)^{1/2} \right] \]

where \( N_c = 2 \left( \frac{2m^*kT}{\hbar^2} \right)^{1/2} \)

\( N_c \) is the effective density of states as if all electrons at conduction band edge \( E_c \). Only limiting assumption is that \( E_c - E_F \gg kT \); if so, result holds for intrinsic and extrinsic.

(Note: \( E_c - E_F \) for N-type, but approx usually still OK.)

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Electron Concentrations

- Simplify: represent states throughout the conduction band by an effective density of states $N_c$ located at $E_c$.

$$n_0 = N_c f(E_c)$$

- If $E_c - E_F > 3kT$ then

$$f(E_c) \approx e^{(E_c - E_F)/kT} \quad \text{(Maxwell-Boltzmann)}$$

- $kT = 0.026$ eV at 300K, so usually OK

$$n_0 = N_c e^{(E_c - E_F)/kT} ; N_c = 2\left[\frac{2}{\pi}\right]^{3/2} m^* kT/\hbar^2$$

- Terned non-degenerate.

- If $E_F$ within $3kT$ of either band edge or inside band $\rightarrow$ degenerate and we need to use the full Fermi function.

Hole Concentrations

- Concentration of holes in valence band

$$p_0 = N_v [1 - f(E_v)]$$

- If $E_F - E_v > 3kT$ then Fermi function reduces to Maxwell Boltzmann and

$$p_0 = N_v \exp\left[ -\frac{(E_F - E_v)}{kT} \right]$$

$$N_v = \left[\frac{2\pi}{m^* kT/\hbar^2}\right]^{3/2}$$

- $p_0 = \int g_v(E) [1-f(E)] dE$

- $1-f(E) = 1 - \left[1 + \exp((E-E_F)/kT)\right]^{-1}$

$$= \frac{\exp(E-E_F)/kT}{1 + \exp((E-E_F)/kT)}$$

$$= 1/[1 + \exp((E-E_F)/kT)] \approx \exp(-(E_F-E)/kT)$$
Ex 4.1 Determine the probability that a quantum state at $E=E_C+kT$ is occupied by an electron, and calculate the electron concentration in GaAs at $T=300$ K if $E_F=E_C-0.25$ eV.

\[
f_F = \exp\left(\frac{-(E-E_F)}{kT}\right) \\
= \exp\left(\frac{-(E_C+kT-E_F)}{kT}\right) \\
= \exp\left(\frac{-(0.25+0.0259)}{0.0259}\right) \\
f_F = 2.36 \times 10^{-5} \\
N_e = N_c \exp\left(\frac{-(E_C-E_F)}{kT}\right) \\
= (4.7 \times 10^{17}) \exp\left[\frac{-0.25}{0.0259}\right] \\
N_e = 3.02 \times 10^{13}$ cm$^3$ (from Table 4.1)

Ex 4.2 (a) Calculate the equilibrium hole concentration in Si at $T=250$ K if $E_F=E_V+0.27$ eV. [$N_v(300K)=1.04 \times 10^{19}$/cm$^3$] (b) What is $p_0(250)$/$p_0(400)$? (See Example 4.2)

(a) $N_v = (1.04 \times 10^{19}) \left(\frac{250}{300}\right)^{3/2}$
\[
= 7.9115 \times 10^{18}$ cm$^{-3}$
\[
kT = (0.0259) \left(\frac{250}{300}\right) = 0.021583$ eV
\[
p_v = N_v \exp\left(\frac{-(E_F-E_v)}{kT}\right)
\[
= (7.9115 \times 10^{18}) \exp\left[\frac{-0.27}{0.021583}\right]
\[
p_v = 2.919 \times 10^{13}$ cm$^{-3}$

(b) Ratio $= \frac{2.919 \times 10^{13}}{6.43 \times 10^{15}} = 4.54 \times 10^{-3}$
Intrinsic Concentrations

- $E_F = E_{Fi}$ for intrinsic material
- $n_0=n_i = N_c \exp[-(E_C-E_{Fi})/kT]$  $p_i = N_v \exp[-(E_F-E_v)/kT]$
- $n_0 p_0$ = constant.
- Plug in above => $n_0 p_0 = n_i^2 = [N_c N_v] \exp[-E_g/2kT]$
- $n_i = [N_c N_v]^{1/2} \exp[-E_g/2kT]$
  $= 2[2\pi kT/h^2]^{3/2}(m_i^* m_p)^{3/2} \exp[-E_g/2kT]$
- Remember $n_0 p_0 = n_i^2$ and $n_i = 1.5 \times 10^{10}$ cm$^{-3}$ for Si
- You'll use both often!
- Also for Si: 6 different conduction band valleys (2 each x,y,z)
  $m^*_a = 6(m_i m_p)^{1/2}$
- Note: $m^*_a$ varies in different sources
- See Appendix F for more detail (complications) on effective mass

Ex 4.3 (a) Calculate the intrinsic carrier concentration in GaAs at $T=400$ K and $T=250$ K, assuming $E_g = 1.42$eV. (b) What is $n_i(400K)/n_i(250K)$?

(a) For $T = 400$K,

$$n_i^2 = (4.7 \times 10^{17}) \left(7 \times 10^{18} \right) \left(\frac{400}{300}\right) \exp\left[-\frac{1.42}{(0.0259)(400/300)}\right]$$

$= 1.081 \times 10^{19}$

$n_i = 3.29 \times 10^9$ cm$^{-3}$

For $T = 250$K,

$$n_i^2 = (4.7 \times 10^{17}) \left(7 \times 10^{18} \right) \left(\frac{250}{300}\right) \exp\left[-\frac{1.42}{(0.0259)(250/300)}\right]$$

$= 5.09 \times 10^7$

$n_i = 7.13 \times 10^5$ cm$^{-3}$

(b)

$$\frac{n_i(400)}{n_i(250)} = \frac{3.288 \times 10^9}{7.135 \times 10^5} = 4.61 \times 10^3$$
Intrinsic concentrations

- In general, if $E_F \neq E_{Fi}$, then re-write these as
  
  $$ n_0 = n_i \exp\left(\frac{E_F - E_{Fi}}{kT}\right) $$
  
  $$ p_0 = p_i \exp\left(\frac{E_{Fi} - E_F}{kT}\right) $$

- Also used frequently and helpful for visualization
- With each problem consider what information you have and what you are looking for.

- Law of Mass Action:
  - $n_0p_0 = n_i^2$
  - Independent of $E_F$, (i.e. of doping)
  - So applies to both intrinsic and extrinsic at equilibrium

\[
T(r) = \frac{J_0}{T(0)} \rightleftharpoons n_i \propto T^{3/2} \exp\left[-\frac{E_g}{2kT}\right]
\]
### Physical constants (@300 K unless noted)

<table>
<thead>
<tr>
<th>Physical constant</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boltzmann's const (k)</td>
<td>$1.38 \times 10^{-23}$ J/K $8.62 \times 10^{-5}$ ev/K</td>
</tr>
<tr>
<td>Planck’s constant (h)</td>
<td>$6.63 \times 10^{-34}$ J-s $4.14 \times 10^{15}$ eV-s</td>
</tr>
<tr>
<td>Room temp value of kT</td>
<td>0.0259 eV</td>
</tr>
<tr>
<td>Electron rest mass ($m_0$)</td>
<td>9.11 $\times 10^{-37}$ kg</td>
</tr>
<tr>
<td>Planck’s constant (h)</td>
<td>$1.60 \times 10^{-19}$ C</td>
</tr>
<tr>
<td>Permittivity of free space ($\varepsilon_0$)</td>
<td>8.85 $\times 10^{-12}$ F/cm</td>
</tr>
<tr>
<td>Density of states</td>
<td>$2.8 \times 10^{19}$ cm$^{-3}$ $1.04 \times 10^{19}$ cm$^{-3}$</td>
</tr>
<tr>
<td>Relative permittivity of silicon ($\varepsilon_r$)</td>
<td>11.9</td>
</tr>
<tr>
<td>Speed of light (c)</td>
<td>$3 \times 10^{10}$ cm/s</td>
</tr>
<tr>
<td>Electron volt (eV)</td>
<td>1.6 $\times 10^{-19}$ J</td>
</tr>
<tr>
<td>Band Gap Si $E_g$</td>
<td>1.11 eV</td>
</tr>
<tr>
<td>Band gap Ge $E_g$</td>
<td>0.67 eV</td>
</tr>
</tbody>
</table>

### Effective density of states & masses

#### Table 4.1 Effective density of states function and density of states effective mass values

<table>
<thead>
<tr>
<th>Material</th>
<th>$N_e$ (cm$^{-3}$)</th>
<th>$N_h$ (cm$^{-3}$)</th>
<th>$m_e^*/m_0$</th>
<th>$m_h^*/m_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Silicon</td>
<td>$2.8 \times 10^{19}$</td>
<td>$1.04 \times 10^{20}$</td>
<td>1.08</td>
<td>0.56</td>
</tr>
<tr>
<td>Gallium arsenide</td>
<td>$4.7 \times 10^{17}$</td>
<td>$7.0 \times 10^{18}$</td>
<td>0.067</td>
<td>0.48</td>
</tr>
<tr>
<td>Germanium</td>
<td>$1.04 \times 10^{19}$</td>
<td>$6.0 \times 10^{18}$</td>
<td>0.55</td>
<td>0.37</td>
</tr>
</tbody>
</table>

See Appendix B for $m^*$ values

#### Table 4.2 Commonly accepted values of $n_i$ at $T = 300$ K

<table>
<thead>
<tr>
<th>Material</th>
<th>$n_i$ (cm$^{-3}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Silicon</td>
<td>$1.5 \times 10^{16}$</td>
</tr>
<tr>
<td>Gallium arsenide</td>
<td>$1.8 \times 10^{10}$</td>
</tr>
<tr>
<td>Germanium</td>
<td>$2.4 \times 10^{13}$</td>
</tr>
</tbody>
</table>
Intrinsic $E_F$ position

\[ n_0 = p_0 = n_i = p_i \]
\[ N_C \exp \left( \frac{(E_C - E_F)}{kT} \right) = N_V \exp \left( \frac{(E_F - E_V)}{kT} \right) \]
\[ \therefore \ln \left( \frac{N_C}{N_V} \right) + \frac{E_F - E_C}{kT} = \ln \left( \frac{N_V}{N_C} \right) + \frac{E_V - E_F}{kT} \]
\[ E_F = \frac{1}{2} (E_C + E_V) + \frac{1}{2} kT \ln \left( \frac{N_V}{N_C} \right) \]
\[ = \frac{1}{2} (E_C + E_V) + \frac{3}{4} kT \ln \left( \frac{m^*}{m_n} \right) \]

i.e. \[ E_F = E_{\text{midgap}} + \frac{3}{4} kT \ln \left( \frac{m^*}{m_n} \right) \] \[ \Rightarrow \text{if } m^* = m_n \]

Ex 4.4 Find the position of $E_F$ at $T=300$ K wrt the band-gap center for (a) GaAs, and (b) Ge.

(a) GaAs

\[ E_{F1} - E_{\text{midgap}} = \frac{3}{4} kT \ln \left( \frac{m^*}{m_n} \right) \]
\[ = \frac{3}{4} \left( 0.0259 \right) \ln \left( \frac{0.48}{0.067} \right) \]
\[ \Rightarrow +38.25 \text{ meV} \]

(b) Ge

\[ E_{F1} - E_{\text{midgap}} = \frac{3}{4} kT \ln \left( \frac{m^*}{m_n} \right) \]
\[ = \frac{3}{4} \left( 0.0259 \right) \ln \left( \frac{0.37}{0.55} \right) \]
\[ \Rightarrow -7.70 \text{ meV} \]
Summary: Fermi function

- Fermi function describes probability of a given state being occupied at temperature $T$.
  \[ f(E) = \{1 + \exp[\frac{(E-E_F)}{kT}]\}^{-1} \]
- For intrinsic semiconductors $E_F$ is mid-gap.
- If $E - E_F > 3kT$ (non-degenerate, with $E_F$ in the forbidden gap) then
  \[ f(E) \approx e^{-\frac{(E-E_F)}{kT}} \]
- Use effective density of states $N_c$ located at $E_c$ or $N_v$ located at $E_v$ (still non-degenerate)
  \[ n_0 = N_c e^{\frac{(E_c-E_F)}{kT}}, \quad N_c = 2\frac{[2\pi m_v^* kT/h^2]^{3/2}}{\epsilon^*} \]
  \[ p_0 = N_v e^{\frac{(E_v-E_F)}{kT}}, \quad N_v = 2\frac{[2\pi m_p^* kT/h^2]^{3/2}}{\epsilon^*} \]