Section 4.3 Order Relations

A binary relation is an *partial* order if it transitive and antisymmetric. If R is a partial order over the set S, we also say, "S is a partially ordered set" or "S is a poset". To emphasize both S and R we write $\langle S, R \rangle$ and call it a poset.

Example. Here are five sample posets.

$\langle \mathbf{N}, \leq \rangle$, $\langle \mathbf{N}, < \rangle$, $\langle \mathbf{N}, \text{divides} \rangle$, $\langle \text{power}(\{a, b, c\}), \subset \rangle$, and

$\langle \text{Steps\_of\_a\_Recipe}, R \rangle$ where $i\ R\ j$ iff step i is done before step j.

Comparability. Let $\langle S, R \rangle$ be a poset. Then we have the following definitions.

1. $x, y \in S$ are *comparable* if $x\ R\ y$ or $y\ R\ x$.
2. If all pairs of elements are comparable, then R is a *total* (or *linear*) order.
3. A *chain* is a set of elements that are comparable to each other.

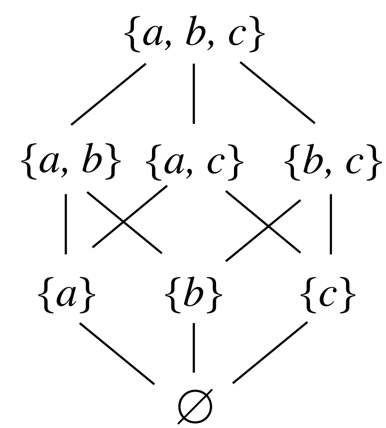
Notation. We'll often use the symbols $<$ and $\leq$ as general symbols for partial orders. If $x < y$, say "x precedes y" or "x is a predecessor of y" or "y is a successor of x."

An element x is an *immediate predecessor* of y or y is an *immediate successor* of x if

$$\{z \mid x < z < y\} = \varnothing.$$

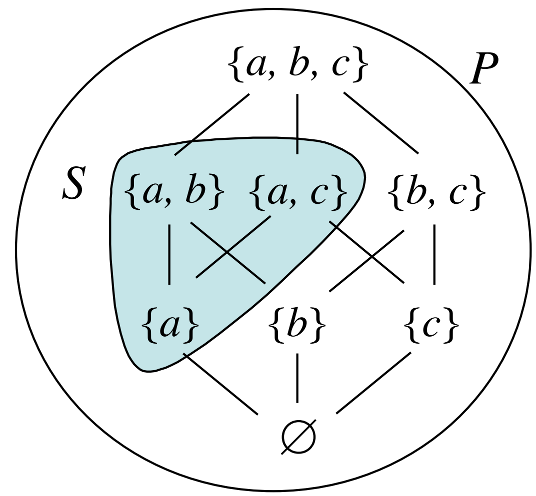
Poset Diagram (Hasse Diagram): A graph representing a poset but with only immediate predecessor edges, and the edges are oriented up from x to y when $x < y$.

Example. The poset $\langle \text{power}(\{a, b, c\}), \subset \rangle$ has the poset diagram shown in the picture.



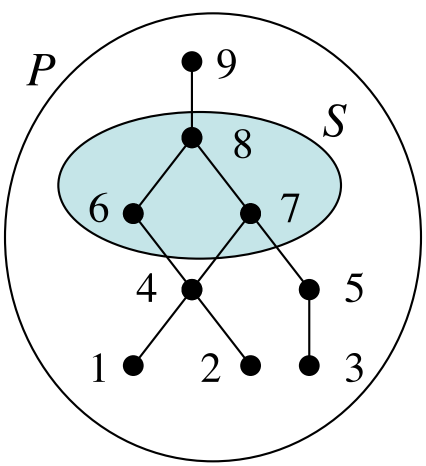
Minima, Maxima, and Bounds. Let S be a subset of a poset P.
An element $x \in S$ is a *minimal element* of S if it has no predecessors in S. A minimal element $x \in S$ is the *least element* of S if $x \leq y$ for all $y \in S$.
An element $x \in P$ is a *lower bound* of S if $x \leq y$ for all $y \in S$. A lower bound x of S is the *greatest lower bound* of S, denoted glb(S), if $y \leq x$ for all lower bounds y of S.
There are corresponding definitions for *maximal element* of S, *greatest element* of S, *upper bound* of S, and *least upper bound* of S, lub(S).

Example. Let P = power($\{a, b, c\}$) shown in the picture and let $S = \{\{a\}, \{a, b\}, \{a, c\}\}$. Then {a} is the only minimal element of S, so $\{a\}$ is the least element of S. The lower bounds of S are $\{a\}$ and $\varnothing$, with glb(S) = $\{a\}$. The maximal elements of S are $\{a, b\}$ and $\{a, c\}$, but there is no greatest element of S. The only upper bound of S is $\{a, b, c\}$, so lub(S) = $\{a, b, c\}$.



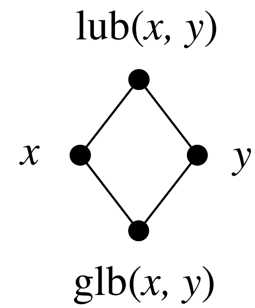
Quiz (2 minutes). Find the minima, maxima, and bounds for the subset S ={6, 7, 8} of the poset P pictured poset diagram.

Solution. The minimal elements of S are 6 and 7, but there is no least element of S. The lower bounds of S are 1, 2, 4 with glb(S) = 4. The only maximal element of S is 8, so 8 is also the greatest element of S. The upper bounds of S are 8 and 9, with lub(S) = 8.



Lattices

A *lattice* is a poset for which every pair of elements has a glb and a lub.

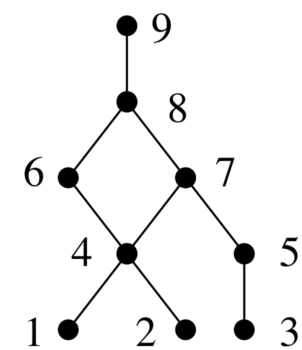


Example. For any set S the poset $\langle$ power(S), $\subset \rangle$ is a lattice because for any sets A and B, we have glb(A, B) = $A \cap B$ and lub(A, B) = $A \cup B$.

Example/Quiz. Is $\langle \{1, 2, 3, 4, 5, 6\}, \mid \rangle$ a lattice?
Answer: No. For example, there is no lub for 2 and 5.
Quiz. Is $\langle \{1, 2, 3, 6, 12\}, \mid \rangle$ a lattice?
Answer: Yes.

Topological Sorting (*Sorting a Poset*) The idea is to output a minimal element and then remove it from the poset diagram and continue the process with the modified poset. The resulting output will always have the property that x is output before y if $x < y$.

Algorithm. A typical algorithm keeps track of the number $p(x)$ of immediate predecessors and the set $s(x)$ of immediate successors of each element x, and the set Sources = $\{x \mid p(x) = 0\}$.



while Sources $\neq \emptyset$ **do**
  Output a source x and remove it from Sources;
  Decrement $p(y)$ for each $y \in s(x)$ and update Sources
od

Example. The poset pictured has several possible topological sorts. The initial sources are 1, 2, 3. If we output 3, then the sources become 1, 2, 5. Two possible topological sorts are:

3, 5, 2, 1, 4, 7, 6, 8, 9 and 1, 2, 4, 6, 3, 5, 7, 8, 9.

Well-Founded Orders

A poset is *well-founded* if every nonempty subset has a minimal element or, equivalently, every descending chain of elements

$$x_1 > x_2 > \ldots$$

is finite. To see the equivalence, notice that if all descending chains are finite, then the last element must be minimal. And, if all nonempty subsets have minimal elements, then you can't find a descending chain that goes forever since it would contradict minimality.

Example. The posets $\langle \mathbf{N}, < \rangle$ and $\langle$ power(finite set), $\subset \rangle$ are well-founded.

Example. The posets $\langle \mathbf{Z}, < \rangle$ and $\langle$ power(infinite set), $\subset \rangle$ are not well-founded. For example, an infinite descending chain in power(**N**) is

$$\mathbf{N} \supset \mathbf{N} - \{0\} \supset \mathbf{N} - \{0, 1\} \supset \mathbf{N} - \{0, 1, \ldots, n\} \supset \ldots$$

Lexicograpic Ordering of $\mathbf{N}^n$

The lexicographic ordering on n-tuples is defined by

$$(x_1, \ldots, x_n) < (y_1, \ldots, y_n) \text{ iff } x_1 < y_1 \text{ or } (x_i = y_i \text{ for } 1 \le i < j \text{ and } x_j < y_j).$$

This ordering is well-founded. Notice that it is also linear.

Example Proof: We'll show the lexicographic ordering on $\mathbf{N}^2 = \mathbf{N} \times \mathbf{N}$ is well-founded. First, notice that any descending chain of the form $(x, y_1) > (x, y_2) > \ldots$ must stop since $y_1 > y_2 > \ldots$ is a finite descending chain in **N**. To continue such a chain the first argument of a pair must decrease at some point. For example, $(x_1, y_1) > (x_2, y_2)$ where $x_1 > x_2$. But there are only finitely many such changes possible because $x_1 > x_2 > \ldots$ is a finite descending chain in **N**. So the lexicographic ordering on $\mathbf{N}^2$ is well-founded. QED

Lexicograpic Ordering of A^*

Let A be an alphabet with some agreed upon ordering. If x and y are strings over A, then $x < y$ iff either x is a proper prefix of y (i.e., $y = xz$ with $z \neq \Lambda$) or x and y have a longest common proper prefix u such that $x = uw$ and $y = uz$, and head(w) < head(z) in A.

NOTE: This is the usual dictionary ordering of strings and it is NOT well-founded, although it is a total order. For example, we have the following infinite descending chain of strings over $\{a, b\}$ where we assume that $a < b$:

$$b > ab > aab > aaab > \ldots$$

Quiz (1 minute). Order all strings of length 3 over $\{a, b\}$ where $a < b$.

Answer: $aaa < aab < aba < abb < baa < bab < bba < bbb$.

Standard Ordering of A^*

This is a well-founded ordering that orders strings by length and uses lexicographic for strings of the same length.

Example. Here is the longest descending chain starting from aaa over $\{a, b\}$ where $a < b$:

$$aaa > bb > ba > ab > aa > b > a > \Lambda.$$

A Simple Construction Technique for Well-Founded Orders

Any function $f : S \rightarrow \mathbf{N}$ defines a well-founded order on S by

$$x < y \text{ iff } f(x) < f(y).$$

Example. Lists are well-founded by length. Binary trees are well-founded by depth, by number of nodes, or by number of leaves. $\mathbf{Z}$ is well-founded by absolute value. Derivations for a grammar are well-founded by length. These orders are nonlinear.

Example. Let $f : \mathbf{Z} \to \mathbf{N}$ be defined by $f(x)$ = if $x \geq 0$ then $2x$ else $-2x - 1$. The ordering on $\mathbf{Z}$ defined by $x < y$ iff $f(x) < f(y)$ is well-founded and linear: $0 < -1 < 1 < -2 < 2 < \ldots$.

Well-Founded Orders for Inductively Defined Sets

If S is an inductively defined set and no two elements are defined in terms of each other, then the following methods can be used to establish a well-founded order for S.

Method A. Define $f : S \to \mathbf{N}$ by $f(b) = 0$ for each basis element b; and if x is defined in terms of $y_1, \ldots, y_n$, then $f(x) = 1 + \max\{f(y_1), \ldots, f(y_n)\}$. Define $x < y$ iff $f(x) < f(y)$.

Method B. The basis elements are the minimal elements of S and if x is defined in terms of $y_1, \ldots, y_n$, then set $y_i < x$ for each i. Now take the transitive closure of the this ordering.

Example. The set $\mathbf{Z}$ can be defined inductively by:

Basis: $0 \in \mathbf{Z}$
Induction: $x \in \mathbf{Z}$ implies $x + 1, x - 1 \in \mathbf{Z}$.

Notice that 1 and –1 are constructed from 0 but 0 is also constructed from –1. So neither technique builds a poset.

Example. The set $\mathbf{N}^2$ can be defined inductively by:

Basis: $(0, 0) \in \mathbf{N}^2$
Induction: $(x, y) \in \mathbf{N}^2$ implies $(x, y + 1), (x + 1, y) \in \mathbf{N}^2$.

In the ordering constructed by method A any pair (x, y) has $n + 2$ immediate successors if $x + y = n$. For example, the the immediate successors of (0, 1) are (0, 2), (1, 1) and (2, 0). In the ordering constructed by method B any pair (x, y) has exactly 2 immediate successors. For example, the the immediate successors of (0, 1) are (0, 2) and (1, 1).